Universidade de Aveiro Departamento de Matemática
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palavras-chave

Espaços de Besov, diferenças, homogeneidade, átomos não suaves, fractais, conjuntos-*h*, espaços-*h*, operadores extensão, laplaciano, problema de Dirichlet.

resumo

Nesta tese são estudados espaços de Besov de suavidade generalizada em espaços euclidianos, numa classe de fractais designados conjuntos-*h* e em estruturas abstractas designadas por espaços-*h*.
Foram obtidas caracterizações e propriedades para estes espaços de funções. Em particular, no caso de espaços de Besov em espaços euclidianos, foram obtidas caracterizações por diferenças e por decomposições em átomos não suaves, foi provada uma propriedade de homogeneidade e foram estudados multiplicadores pontuais.
Para espaços de Besov em conjuntos-*h* foi obtida uma caracterização por decomposições em átomos não suaves e foi construído um operador extensão. Com o recurso a cartas, os resultados obtidos para estes espaços de funções em fractais foram aplicados para definir e trabalhar com espaços de Besov de suavidade generalizada em estruturas abstractas.
Nesta tese foi também estudado o laplaciano fractal, considerado a actuar em espaços de Besov de suavidade generalizada em domínios que contêm um conjunto-*h* fractal. Foram obtidos resultados no contexto de teoria espectral para este operador e foi estudado, à custa deste operador, um problema de Dirichlet fractal no contexto de conjuntos-*h*.

keywords

Besov spaces, differences, homogeneity, non-smooth atoms, fractals, *h*-sets, *h*-spaces, extension operators, Laplacian, Dirichlet problem.

abstract

In this thesis Besov spaces of generalised smoothness on Euclidean spaces, on a class of fractals called *h*-sets and on abstract structures called *h*-spaces are studied. Characterisations and properties for these function spaces were obtained. In particular, in the case of Besov spaces on Euclidean spaces, characterisations by differences and non-smooth atomic decompositions were obtained, a homogeneity property was proved and pointwise multipliers were studied.

For Besov spaces on *h*-sets a characterisation by non-smooth atomic decompositions was proved and an extension operator was constructed. Using convenient charts, the results obtained for these function spaces were applied to define and work with Besov spaces of generalised smoothness on abstract structures.

In this thesis the fractal Laplacian was also studied, considered acting in Besov spaces of generalised smoothness on domains which contain an *h*-set. We obtained results for this operator in the context of spectral theory and we applied the results on this operator to study a fractal Dirichlet problem in the context of *h*-sets.

Contents

Introduction

In this thesis we study Besov spaces of generalised smoothness on the Euclidean space and on some classes of structures called h-spaces and h-sets.

Besov spaces of generalised smoothness on $\mathbb{R}^n$ extend the classic Besov spaces usually denoted by $B^s_{p,q}(\mathbb{R}^n)$. For systematic studies of classic Besov spaces we refer to the books by Triebel [Tri83] and [Tri92b]. The first chapters both of [Tri92b] and [Tri06] are surveys (entitled *How to measure smoothness*) where one finds the history of these spaces.
Function spaces of generalised smoothness have been studied by many authors, by different approaches and considering different degrees of generality. We refer to Goldman (cf. [Gol76]) for an approach using modulus of continuity, and to Kalyabin for an approach using approximation by entire functions of exponential type (cf. [KL87]). The survey [KL87] covers a great part of the literature of what was done in this context up to the eighties. In [Mer84] and [CF88], Merucci and Cobos and Fernandez, respectively, studied function spaces of generalised smoothness which are obtained as the interpolation space of classic spaces. In [FL06] Farkas and Leopold considered the general setting of quasi-Banach spaces defined by a Fourier analytical approach based on a suitable resolution of unity on the Fourier side and a suitable weighted summation of the resulting parts. In this work we follow this approach. We also refer to [FL06] for more detailed information and references related to the history of function spaces of generalised smoothness. In [Bri01] by Bricchi and [Mou01b, Mou07] by Moura, some characterisations of function spaces of generalised smoothness, which will be applied in this work, were obtained.

As we have already mentioned, we consider in this thesis function spaces on structures called h-sets and h-spaces. Let us give a brief description of these classes of structures.

An *h-space* $X = (X, \varrho, \mu)$ is a compact set X with respect to a quasi-metric ϱ and endowed with a Borel measure μ such that the measure of a ball of radius r is equivalent to $h(r)$, for some suitable function h, i.e.,

$$\mu(B(x,r)) \sim h(r) \quad \text{for all } x \in X \text{ and } 0 < r \leq 1. \tag{0.0.1}$$

If we have in particular $X \subset \mathbb{R}^n$ and ϱ the usual Euclidean metric, we use a different notation: we denote the set by the letter Γ and the metric by ϱ_n. We say in this case that $\Gamma = (\Gamma, \varrho_n, \mu)$ is an *h-set*. In [Bri01] and [Bri04] Bricchi characterised the class of admitted functions h. There he also introduced and studied Besov spaces of generalised smoothness on these fractal sets. We also refer to [KZ06] for characterisations of these function spaces, which will be used later in this thesis. In the particular case where $h(r) = r^d\psi(r)$, for a number $0 < d \leq n$ and a monotone function $\psi : (0, 1] \to \mathbb{R}^+$ such that $\psi(2^{-j}) \sim \psi(2^{-2j})$, $j \in \mathbb{N}_0$, then we say that Γ is a *(d, ψ)-set*. If, additionally, $\psi \sim 1$, we say that Γ is a *d-set*. Function spaces on d-sets have been studied with several methods. We refer to Jonsson and Wallin (cf. [JW84]) and Triebel (cf. [Tri97] and [Tri01]). For the study of function spaces on (d, ψ)-sets we refer to [Mou01b].

Another particular class of h-spaces that have been considered are *d-spaces*. In these cases, we also have $h(r) = r^d$, for some $d > 0$, but now for (abstract) quasi-metric spaces X. Function spaces on this kind of spaces have been studied by Han and Yang, namely in [HY02], where *approximations to the identity* are used to define the spaces. We refer to [Tri06, 1.17.4], where several references are given and a comparison between this approach to function spaces on quasi-metric spaces and the description of spaces $B^s_{p,q}(\mathbb{R}^n)$ in terms of local means is made.

In [Tri05] Triebel presented a different approach to define Besov spaces on d-spaces, using *snowflaked transforms* and *Euclidean charts*, which allow to transfer the study of function spaces on quasi-metric spaces to spaces on fractal sets in some $\mathbb{R}^n$. Results involving applications in function spaces on d-spaces were also presented there, obtained by making use of what is known about d-sets. It was also proved that, in some cases, the

Besov spaces defined this way are the same spaces as introduced by Han and Yang.

In this thesis we consider Besov spaces of generalised smoothness on h-spaces, which includes d-spaces, d-sets and h-sets.

We follow Triebel's approach and so we study first function spaces on $\mathbb{R}^n$, then, via traces, on h-sets, and finally, using charts, on h-spaces.

In Chapter 1 we fix some notation. We also present a collection of notions and properties related to sequences and functions that often appear in the context of function spaces of generalised smoothness. Chapter 2 is devoted to Besov spaces on $\mathbb{R}^n$. We deal with characterisations by differences, by quarkonial decompositions and by smooth and non-smooth atomic decompositions. We also prove a homogenetity property for classic and general Besov spaces. This property was already known for classic Triebel-Lizorkin spaces (cf. [Tri01, Corollary 5.16, p. 66]).

In Chapter 3 we consider Besov spaces on h-sets, defined by traces. For such spaces we also deal with smooth and non-smooth atomic decompositions. Furthermore, we consider a subclass of these spaces following an approach analogous to Jonsson and Wallin's approach to define function spaces on d-sets.

In Chapter 4 we define Besov spaces of generalised smoothness on h-spaces. We describe the techniques and tools used in this construction, namely snowflaked transforms and charts. We apply the results obtained for spaces on fractal h-sets to conclude for these abstract quasi-metric spaces.

In Chapter 5, the last chapter of this thesis, we study spectral theory for the fractal Laplacian and we study a Dirichlet problem in the context of h-sets. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ such that $\Gamma \subset \Omega$ and $-\Delta$ be the Dirichlet Laplacian in Ω. We study the operator

$$B := (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma,$$

acting in convenient function spaces in Ω, where $(-\Delta)^{-1}$ is the inverse of the Dirichlet Laplacian in Ω and

$$\mathrm{tr}^\Gamma = \mathrm{id}^\Gamma \circ \mathrm{tr}_\Gamma,$$

where tr_Γ is an extension of the operator

$$\varphi \to \varphi_{|\Gamma}, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n)$$

and id^Γ, which will be formally defined later, identifies elements of $L_p(\Gamma)$ with tempered distributions.

The operator B was already studied in the case where Γ is a d-set by Triebel in [Tri97, Chapter 5] and [Tri01, Chapter 3]. The case where Γ is a (d,ψ)-set was studied by Edmunds and Triebel in [ET98, ET99] and by Moura in [Mou01b]. As it was mentioned in these works, in the case $n = 2$ the operator B has physical relevance: it describes the vibration of a drum where the whole mass is distributed on Γ. This is the reason why the study of this subject is usually called the *fractal drum* problem. So we study the *fractal drum* problem in the context of h-sets, extending the results for d-sets and (d,ψ)-sets. Applying the results obtained for the operator B we also prove the existence of a solution for the fractal Dirichlet problem for h-sets. This problem was studied by Triebel for the case of d-sets (cf. [Tri01, Chapter 3, Section 20]).

With the purpose of studying the operator B some other interesting results for Besov spaces of generalised smoothness were obtained: a result on pointwise multipliers for these spaces and the existence of a universal extension operator acting from Besov spaces on a class of domains into corresponding function spaces on $\mathbb{R}^n$. In the proof of some of these results we used interpolation (with a function parameter), an important tool in the theory of function spaces.

A great part of the results obtained in this thesis are published in the papers [CLT07] and [CL09].

Chapter 1

Preliminaries

In this chapter we fix some notation. In the first section we present some general notation. In the second one we present a list of concepts and properties related to sequences and functions that we will apply in the whole work in connection with the idea of function spaces of *generalised smoothness*. In Section 3 some basic notation in measure theory is fixed and in Section 4 Hausdorff measures and Hausdorff dimensions are defined.

1.1 General Notation

We introduce some standard notation and useful definitions. We write $\mathbb{N}$ for the set of all natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all integers, real and complex numbers, respectively. As usually, $\mathbb{R}^n$, $n \in \mathbb{N}$, stands for the n-dimensional real Euclidean space and, given $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $|x|$ stands for the Euclidean norm of x. If $x, y \in \mathbb{R}^n$ then $x \cdot y$ denotes the scalar product of x and y, i.e., $x \cdot y = x_1 y_1 + ... + x_n y_n$. We denote by $\mathbb{Z}^n$, where $n \in \mathbb{N}$, the lattice of all points $m = (m_1, ..., m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$, $j = 1, ..., n$. We write $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, to represent the collection of all multi-indices $\alpha = (\alpha_1, ..., \alpha_n)$ with $\alpha_j \in \mathbb{N}_0$, $j = 1, ..., n$. For $\alpha \in {\mathbb{N}_0}^n$, $|\alpha| := \alpha_1 + ... + \alpha_n$, $\alpha! = \alpha_1!...\alpha_n!$ and $D^\alpha := \partial^{|\alpha|}/(\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n})$ denotes the classic or weak partial derivative of order α. For $x = (x_1, .., x_n) \in \mathbb{R}^n$ and $\alpha \in {\mathbb{N}_0}^n$, $x^\alpha := x_1^{\alpha_1} ... x_n^{\alpha_n}$.

If there is no additional information, when we speak about "functions" we are considering complex-valued functions and references with respect to measurability and integrability should be understood in the Lebesgue sense. We write supp to denote the support of a function, a distribution or a measure.

If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for its Lebesgue measure in $\mathbb{R}^n$ and χ_E denotes the characteristic function over E.

An open ball with center in $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(x,r)$ or $B_r(x)$. If $E \subset \mathbb{R}^n$, then ∂E denotes the boundary of E and $\overline{E}$ stands for the closure of E.

If $E \subset \mathbb{R}^n$ and $r > 0$ we denote by E_δ the r-neighborhood of E, i.e., the collection of all points x in $\mathbb{R}^n$ such that there is $y \in E$ for which $|x - y| < r$.

We use the symbol "$\lesssim$" in

$$a_k \lesssim b_k \quad \text{or} \quad \phi(r) \lesssim \psi(r)$$

always to mean that there is a positive number c_1 such that

$$a_k \leq c_1 b_k \quad \text{or} \quad \phi(r) \leq c_1 \psi(r)$$

for all admitted values of the discrete variable k or the continuous variable r, where $(a_k)_k$, $(b_k)_k$ are non-negative sequences and ϕ, ψ are non-negative functions. We use the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \phi(r) \sim \psi(r)$$

for

$$a_k \lesssim b_k \quad \text{and} \quad b_k \lesssim a_k \qquad \text{or} \qquad \phi(r) \lesssim \psi(r) \quad \text{and} \quad \psi(r) \lesssim \phi(r).$$

We represent by $[\cdot]$ the integer-part function. We use the standard abbreviation

$$a_+ = \max\{0, a\}, \quad a \in \mathbb{R}.$$

In what follows log is always taken with respect to the base 2. We use the symbol $\hookrightarrow$ to represent a continuous embedding between (quasi-)normed spaces.

We write $C^\infty(\mathbb{R}^n)$ to denote the class of all infinitely differentiable functions and by $C_0^\infty(\mathbb{R}^n)$ the collection of all functions in $C^\infty(\mathbb{R}^n)$ with compact support.

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all rapidly decreasing functions in $C^\infty(\mathbb{R}^n)$, equipped with the usual topology, and by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, the space of all tempered distributions on $\mathbb{R}^n$.

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\mathcal{F}\varphi$, or $\widehat{\varphi}$, represents the Fourier transform of φ, i.e.,

$$(\mathcal{F}\varphi)(\xi) := \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx, \quad \xi \in \mathbb{R}^n,$$

and $\mathcal{F}^{-1}\varphi$, or $\check{\varphi}$, represents the inverse Fourier transform of φ,

$$(\mathcal{F}^{-1}\varphi)(\xi) := \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(x)dx, \quad \xi \in \mathbb{R}^n.$$

Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we will use the notation

$$\varphi(D)f := \mathcal{F}^{-1}(\varphi\mathcal{F}f).$$

As usual, "domain" stands for "open set". If Ω is a domain in $\mathbb{R}^n$ then $L_p(\Omega)$ denotes the collection of all complex-valued Lebesgue measurable functions in Ω such that

$$\|f|L_p(\Omega)\| := \Big(\int_\Omega |f(x)|^p dx\Big)^{\frac{1}{p}}$$

(with the usual modification $\operatorname{ess\,sup}_{x\in\Omega}|f(x)|$ if $p=\infty$) is finite.

Furthermore, if $0<q\le\infty$ then ℓ_q is the collection of all complex sequences $a=(a_j)_{j\in\mathbb{N}_0}$ such that

$$\|a|\ell_q\| := \Big(\sum_{j=0}^\infty |a_j|^q\Big)^{1/q}$$

(with the standard adaptation $\sup_{j\in\mathbb{N}_0}|a_j|$ if $q=\infty$) is finite.

If $0<p,q\le\infty$ and $(f_j)_{j\in\mathbb{N}_0}$ is a sequence of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$, then

$$\|(f_j)_{j\in\mathbb{N}_0}|\ell_q(L_p)\| := \Big(\sum_{j=0}^\infty \|f_j|L_p\|^q\Big)^{1/q}$$

with the appropriate modification if $q=\infty$ and

$$\|(f_j)_{j\in\mathbb{N}_0}|L_p(\ell_q)\| := \Big(\int_{\mathbb{R}^n} \|f_j(x)|\ell_q\|^p dx\Big)^{1/p}$$

with the appropriate modification if $p = \infty$.
If Ω is a domain in $\mathbb{R}^n$ then $C(\overline{\Omega})$ denotes the collection of all complex-valued continuous functions on $\overline{\Omega}$ and $C^\infty(\overline{\Omega})$ is the collection of all functions f such that $D^\alpha f \in C(\overline{\Omega})$, for all $\alpha \in {\mathbb{N}_0}^n$.
If Ω is a domain, $\mathcal{D}(\Omega)$ is the collection of all compactly supported complex-valued C^∞ functions in Ω and $\mathcal{D}'(\Omega)$ stands for the dual space of all distributions on Ω.
If X and Y denote quasi-normed spaces, we represent by $L(X, Y)$ the space of all bounded linear maps from X to Y. If T is an operator we denote by $N(T)$ the null-space of T. If S denotes a subspace of a space with a scalar product then we write $S^\perp$ to represent the orthogonal complement of S.
All unimportant constants are denoted by c and may sometimes represent different constants in a single chain of inequalities. Sometimes we distinguish them using different representations (c_1 and c', for instance). For convenience sometimes it is emphasized the dependence of the constants in some parameters. For example, we shall write c_ε with the understanding that the constant depends on the parameter ε.
Further notation will be presented whenever is introduced.

1.2 Sequences, indices and functions

As we described in the Introduction, in our work we consider Besov spaces of generalised smoothness. In this context some classes of sequences are considered. In the results, the restrictions on these sequences are given in terms of convenient indices. Moreover, sometimes it is convenient to associate these sequences to appropriate functions. In this section we present some definitions, notation and properties on these sequences, indices and functions that will be applied on the next chapters.

Definition 1.2.1. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers. We say that σ is an admissible sequence if there are positive constants d_0, d_1 such that*

$$d_0\sigma_j \leq \sigma_{j+1} \leq d_1\sigma_j, \quad j \in \mathbb{N}_0. \tag{1.2.1}$$

We introduce two particular kinds of admissible sequences that we will use throughout the next chapters.

Example 1.2.2. *(i) We will denote by* (s) *the (admissible) sequences defined by*

$$(s) := (2^{js})_{j\in\mathbb{N}_0}, \quad s \in \mathbb{R}. \tag{1.2.2}$$

(ii) Let $\psi : (0,1] \to \mathbb{R}$ *be a positive monotone function such that* $\psi(2^{-2j}) \sim \psi(2^{-j})$*, for all* $j \in \mathbb{N}_0$*. We will denote by* (s,ψ) *the sequences*

$$(s,\psi) := (2^{js}\psi(2^{-j}))_{j\in\mathbb{N}_0}, \quad s \in \mathbb{R},$$

which are also admissible sequences.

Notation 1.2.3. *Let* $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ *and* $\beta = (\beta_j)_{j\in\mathbb{N}_0}$ *be admissible sequences and* $\alpha \in \mathbb{R}$*. In what follows we denote by* σ^α *and* $\sigma\beta$ *the sequences defined by*

$$\sigma^\alpha = (\sigma_j^\alpha)_{j\in\mathbb{N}_0} \quad \textit{and} \quad \sigma\beta = (\sigma_j\beta_j)_{j\in\mathbb{N}_0}, \tag{1.2.3}$$

respectively. It can be easily verified that both σ^α *and* $\sigma\beta$ *are admissible sequences.*

As we mentioned previously, in most results presented in this work the restrictions on the sequences are given in terms of convenient indices of the sequences. In the context of generalised smoothness these indices play in some sense the role of the regularity index usually denoted by s in the case of function spaces with classical smoothness. We will use the indices that we describe next.

Definition 1.2.4. *Let* $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ *be an admissible sequence and*

$$\underline{\sigma}_j := \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \quad \textit{and} \quad \overline{\sigma}_j := \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k}, \quad j \in \mathbb{N}_0.$$

The lower and upper Boyd indices of the sequence σ *are defined, respectively, by*

$$\underline{s}(\sigma) := \lim_{j\to\infty} \frac{\log \underline{\sigma}_j}{j} \quad \textit{and} \quad \overline{s}(\sigma) := \lim_{j\to\infty} \frac{\log \overline{\sigma}_j}{j}.$$

Remark 1.2.5. *For an admissible sequence σ, the sequence $(\log \overline{\sigma}_j)_{j\in\mathbb{N}_0}$ is sub-additive. This justifies the definition of $\overline{s}(\sigma)$. As $\log \underline{\sigma}_j = -\log(\overline{\sigma^{-1}})_j$, $\underline{s}(\sigma)$ also makes sense.*
We remark that if σ and β are admissible sequences such that $\sigma \sim \beta$, then their Boyd indices coincide.
We will apply frequently the following property: for each $\delta > 0$ there are two positive constants $c_1 = c_1(\delta)$ and $c_2 = c_2(\delta)$ such that for all $j, k \in \mathbb{N}_0$,

$$c_1 2^{(\underline{s}(\sigma)-\delta)j} \leq \frac{\sigma_{j+k}}{\sigma_k} \leq c_2 2^{(\overline{s}(\sigma)+\delta)j}. \tag{1.2.4}$$

Remark 1.2.6. *In the context of generalised smoothness Bricchi (cf. [Bri01]) considered*

$$\underline{\mathfrak{s}}(\sigma) = \liminf_{j\to\infty} \log\left(\frac{\sigma_{j+1}}{\sigma_j}\right), \quad \overline{\mathfrak{s}}(\sigma) = \limsup_{j\to\infty} \log\left(\frac{\sigma_{j+1}}{\sigma_j}\right).$$

If σ is an admissible sequence, then by (1.2.1) *it follows immediately that both $\underline{\mathfrak{s}}(\sigma)$ and $\overline{\mathfrak{s}}(\sigma)$ are well-defined and finite. The indices $\underline{s}(\sigma)$ and $\underline{\mathfrak{s}}(\sigma)$ [respectively $\overline{s}(\sigma)$ and $\overline{\mathfrak{s}}(\sigma)$] may not coincide. For example, for the sequence*

$$\sigma = \begin{cases} 2^{2\alpha}, & j \text{ even}, \\ 2^{\alpha}, & j \text{ odd}, \end{cases}$$

for a fixed $\alpha > 0$, one can easily check that $\underline{s}(\sigma) = \overline{s}(\sigma) = 0$, $\underline{\mathfrak{s}}(\sigma) = -\alpha$ and $\overline{\mathfrak{s}}(\sigma) = \alpha$.
In [Bri01], the results are stated considering conditions on $\underline{\mathfrak{s}}(\sigma)$ and $\overline{\mathfrak{s}}(\sigma)$. These conditions were considered in order to get estimations of the same kind of (1.2.4) *with $j = 1$ or $k = 0$. Most of them can be adapted and written using $\underline{s}(\sigma)$ and $\overline{s}(\sigma)$. We will do that adaptation whenever we quote Bricchi's work.*

In the next proposition we present some properties which follow immediately from Definition 1.2.4. We use here the notation introduced in (1.2.2).

Proposition 1.2.7. *Let σ and β be admissible sequences and $\alpha \in \mathbb{R}$. Then*

(i) $\underline{s}(\sigma\beta) \geq \underline{s}(\sigma) + \underline{s}(\beta)$ *and* $\overline{s}(\sigma\beta) \leq \overline{s}(\sigma) + \overline{s}(\beta)$;

(ii) $\underline{s}((\alpha)\sigma) = \alpha + \underline{s}(\sigma)$ *and* $\overline{s}((\alpha)\sigma) = \alpha + \overline{s}(\sigma)$;

(iii) $\underline{s}(\sigma^\alpha) = \alpha\underline{s}(\sigma)$ *and* $\overline{s}(\sigma^\alpha) = \alpha\overline{s}(\sigma)$, *if* $\alpha > 0$;

(iv) $\underline{s}(\sigma^\alpha) = \alpha\overline{s}(\sigma)$ *and* $\overline{s}(\sigma^\alpha) = \alpha\underline{s}(\sigma)$, *if* $\alpha < 0$.

Let us present some classes of functions that will appear in the next chapters, associated to admissible sequences.

Definition 1.2.8. *(i) We denote by* $\mathcal{A}$ *the collection of all continuous functions* $f : (0,\infty) \to (0,\infty)$ *such that, for any* $b > 0$,

$$f(bz) \sim f(z), \quad z > 0.$$

If f *belongs to* $\mathcal{A}$ *we say that* f *is an admissible function. (ii) We denote by* $\mathcal{B}$ *the collection of all continuous functions* f, $f : \mathbb{R}^+ \to \mathbb{R}^+$, *such that* $f(1) = 1$ *and*

$$\overline{f}(t) := \sup_{s>0} \frac{f(st)}{f(s)} < \infty, \quad \textit{for all} \quad t > 0. \tag{1.2.5}$$

(iii) We denote by $\mathbf{B}$ *the collection of all* $f \in \mathcal{B}$ *such that*

$$f(t^{-1}) = f(t)^{-1}, \qquad \textit{for all} \quad t > 0.$$

Remark 1.2.9. *Clearly* $\mathbf{B} \subset \mathcal{B} \subset \mathcal{A}$. *The class* $\mathcal{A}$ *was considered in [CF06] and [Mou07]. In [Mer84] and [CF88] in the results on interpolation the class* $\mathcal{B}$ *was considered. In [BM03], for technical reasons, an additional condition was considered originating the class that we denote by* $\mathbf{B}$.

We fix some more notation.

Convention 1.2.10. *Hereafter, by* N *we will always denote a sequence* $N = (N_j)_{j\in\mathbb{N}_0}$ *of real positive numbers such that there exist two numbers* $1 < \lambda_0 \le \lambda_1$ *with*

$$\lambda_0 N_j \le N_{j+1} \le \lambda_1 N_j, \quad j \in \mathbb{N}_0.$$

Remark 1.2.11. *In particular,* N *is admissible and is a so-called strongly increasing sequence (cf. [FL06]). For such* N *there exists a number* $l_0 \in \mathbb{N}$ *such that*

$$2N_j \le N_k \ \textit{ for any } \ j,k \ \textit{ such that } \ j + l_0 \le k.$$

This is true if we choose, for instance, $l_0 \in \mathbb{N}_0$ such that

$$\lambda_0^{l_0} \geq 2$$

holds. We fix such an l_0 in what follows.

In some approaches for the study of Besov spaces of generalised smoothness one needs to associate the sequences to functions in these classes. We fix now how this will be done.

Definition 1.2.12. *Let σ be an admissible sequence. We denote by $\mathcal{A}_{\{\sigma,N\}}$ [respectively $\mathcal{B}_{\{\sigma,N\}}$ or $\mathbf{B}_{\{\sigma,N\}}$] the collection of all functions Λ in $\mathcal{A}$ [respectively $\mathcal{B}$ or $\mathbf{B}$] such that*

$$\Lambda(z) \sim \sigma_j \quad \textit{for all} \quad z \in [N_j, N_{j+1}], \quad \textit{and} \quad j \in \mathbb{N}_0.$$

If $N_j = 2^j$, $j \in \mathbb{N}_0$, we shall simply write $\mathcal{A}_\sigma$ [respectively $\mathcal{B}_\sigma$ or $\mathbf{B}_\sigma$]. In this case, if Λ belongs to $\mathcal{A}_\sigma$ we say that Λ is an admissible function associated to σ.

Remark 1.2.13. *Let $\Lambda \in \mathcal{A}_{\{\sigma,N\}}$ and $c_0, c_1 > 0$. Then*

$$\Lambda(x) \sim \sigma_j, \quad \textit{for all} \quad x \quad \textit{such that} \quad x \in [c_0 N_j, c_1 N_j], \quad j \in \mathbb{N}_0,$$

where the equivalence constants are independent of x and $j \in \mathbb{N}_0$. This property is used several times along this thesis.

The examples we present next guarantee that for all admissible sequence σ the classes in Definition 1.2.12 are non-empty.

Example 1.2.14. *Let $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ be an admissible sequence.*
(i) The function

$$\Lambda(z) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{N_{j+1} - N_j}(z - N_j) + \sigma_j & , z \in [N_j, N_{j+1}), j \in \mathbb{N}_0 \\ \\ \sigma_0 & , z \in (0, N_0), \end{cases}$$

(cf. [CF06, Example 2.3, p. 326]) belongs to $\mathcal{A}_{\{\sigma,N\}}$.

(ii) *The function* $\Lambda : (0, \infty) \to (0, \infty)$ *defined by*

$$\Lambda(z) = \begin{cases} \sigma_0^{-1}\big((\sigma_{j+1} - \sigma_j)(2^{-j}z - 1) + \sigma_j\big) & , z \in [2^j, 2^{j+1}), j \in \mathbb{N}_0 \\ \\ \Lambda(z^{-1})^{-1} & , z \in (0,1) \end{cases}$$

(cf. [BM03, p. 386]) belongs to $\mathbf{B}_\sigma$.

We will also deal frequently with the following particular kind of admissible sequences.

Proposition 1.2.15. *Let* $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ *be an admissible sequence and* $\Lambda \in \mathcal{A}_\sigma$. *Let* $\alpha > 0$. *Then*

$$\sigma_\alpha = (\sigma_{\alpha,j})_{j\in\mathbb{N}_0}, \quad \sigma_{\alpha,j} := \Lambda(2^{\alpha j}) \tag{1.2.6}$$

is an admissible sequence. Furthermore,

$$\underline{s}(\sigma_\alpha) = \alpha \underline{s}(\sigma) \quad \textit{and} \quad \overline{s}(\sigma_\alpha) = \alpha \overline{s}(\sigma). \tag{1.2.7}$$

Let $k \in \mathbb{N}_0$. *The sequence*

$$T_k(\sigma) := (\sigma_{j+k})_{j\in\mathbb{N}_0}$$

is admissible and

$$\underline{s}(T_k(\sigma)) = \underline{s}(\sigma) \quad \textit{and} \quad \overline{s}(T_k(\sigma)) = \overline{s}(\sigma). \tag{1.2.8}$$

Proof. *Step 1.* We prove (1.2.7) for $0 < \alpha \le 1$, taking advantage of the fact that, for such α,

$$\{[\alpha k] : k \in \mathbb{N}_0\} = \mathbb{N}_0. \tag{1.2.9}$$

Let $\gamma = \sigma_\alpha$. One can easily see that

$$\frac{\gamma_{j+k}}{\gamma_k} \sim \frac{\sigma_{[\alpha k]+[\alpha j]}}{\sigma_{[\alpha k]}}, \quad \text{for all} \quad j,k \in \mathbb{N}_0. \tag{1.2.10}$$

By (1.2.9) and (1.2.10),

$$\overline{\gamma}_j \sim \overline{\sigma}_{[\alpha j]} \quad \text{and} \quad \underline{\gamma}_j \sim \underline{\sigma}_{[\alpha j]}, \quad j \in \mathbb{N}_0.$$

Hence

$$\underline{s}(\gamma) = \lim_{j\to\infty} \frac{\log \underline{\sigma}_{[\alpha j]}}{j} = \alpha \lim_{j\to\infty} \frac{\log \underline{\sigma}_{[\alpha j]}}{[\alpha j]} \lim_{j\to\infty} \frac{[\alpha j]}{\alpha j} = \alpha \underline{s}(\sigma).$$

Analogously $\overline{s}(\gamma) = \alpha\overline{s}(\sigma)$.

Step 2. Let $\alpha > 1$ and $\gamma = \sigma_\alpha$. Then $\sigma \sim \gamma_{\alpha^{-1}}$ and so, by Remark 1.2.5 and Step 1,

$$\underline{s}(\sigma) = \frac{1}{\alpha}\underline{s}(\gamma) \quad \text{and} \quad \overline{s}(\sigma) = \frac{1}{\alpha}\overline{s}(\gamma).$$

Step 3. We prove (1.2.8) for the lower Boyd index. We fix $k \in \mathbb{N}_0$ and consider $\beta = T_k(\sigma)$. For all $j \in \mathbb{N}_0$,

$$\underline{\beta}_j = \inf_{t\in\mathbb{N}_0} \frac{\beta_{j+t}}{\beta_t} = \inf_{t\in\mathbb{N}_0} \frac{\sigma_{j+k+t}}{\sigma_{k+t}} \geq \underline{\sigma}_j.$$

So $\underline{s}(\beta) \geq \underline{s}(\sigma)$.

For all $j \in \mathbb{N}_0$,

$$\underline{\sigma}_j = \inf\left\{\inf_{t\geq k} \frac{\sigma_{j+t}}{\sigma_t}, \inf_{0\leq t<k} \frac{\sigma_{j+t}}{\sigma_t}\right\} = \inf\left\{\underline{\beta}_j, \inf_{0\leq t<k} \frac{\sigma_{j+t}}{\sigma_t}\right\}.$$

Let $\delta > 0$. Applying (1.2.4),

$$\inf_{0\leq t<k} \frac{\sigma_{j+t}}{\sigma_t} = \inf_{0\leq t<k} \left(\frac{\sigma_{j+t}}{\sigma_{j+k+t}} \cdot \frac{\sigma_{j+k+t}}{\sigma_{k+t}} \cdot \frac{\sigma_{k+t}}{\sigma_t}\right) \geq c2^{-(\overline{s}(\sigma)+\delta)k} 2^{(\underline{s}(\sigma)-\delta)k} \inf_{0\leq t<k} \frac{\beta_{j+t}}{\beta_t}$$

Hence, for all $j \in \mathbb{N}_0$,

$$\underline{\sigma}_j \geq \min\{1, c2^{-(\overline{s}(\sigma)+\delta)k} 2^{(\underline{s}(\sigma)-\delta)k}\} \underline{\beta}_j,$$

and then $\underline{s}(\beta) \leq \underline{s}(\sigma)$. □

Remark 1.2.16. *It follows immediately that, in the conditions of Proposition 1.2.15, the sequence $T_k(\sigma_\alpha)$, given by*

$$T_k(\sigma_\alpha) = (\sigma_{\alpha,j+k})_{j\in\mathbb{N}_0} = (\Lambda(2^{\alpha(j+k)}))_{j\in\mathbb{N}_0}, \tag{1.2.11}$$

is admissible and its Boyd indices can be expressed by means of the corresponding indices of σ, i.e.,

$$\underline{s}(T_k(\sigma_\alpha)) = \alpha\underline{s}(\sigma) \quad \textit{and} \quad \overline{s}(T_k(\sigma_\alpha)) = \alpha\overline{s}(\sigma).$$

For $N_\alpha = (2^{j\alpha})_{j\in\mathbb{N}_0}$, the function $\Lambda(2^{\alpha k}\cdot)$ belongs to $\mathcal{A}_{\{T_k(\sigma_\alpha),N_\alpha\}}$.

Remark 1.2.17. *The notation σ_α, denoting a sequence as in* (1.2.6), *should not be confused with σ_j, denoting a term of the sequence σ. The distinction follows clearly from the context,* *but we make here the convention that a σ with an index will always denote a sequence as in* (1.2.6) *whenever the index can potentially assume non-integer values. Nevertheless, in the case when a sequence is named by means of the letter N as in N_α, we reserve the special meaning $(2^{\alpha j})_{j\in\mathbb{N}_0}$ to it, which will be recalled whenever deemed necessary.*

In the works where functions in the classes $\mathcal{B}$ and $\mathbf{B}$ (presented in Definition 1.2.8) are used, Boyd indices of such functions are also considered. We refer to [Mer84, CF88, BM03], for example. These indices are defined as follows.

Definition 1.2.18. *Let $f \in \mathcal{B}$. The lower and upper Boyd indices of f are defined, respectively, by*

$$\underline{S}(f) := \lim_{t\to 0} \frac{\log \overline{f}(t)}{\log t} \qquad \textit{and} \qquad \overline{S}(f) := \lim_{t\to\infty} \frac{\log \overline{f}(t)}{\log t}$$

Remark 1.2.19. *The function $\overline{f}$ defined in* (1.2.5) *is submultiplicative and Lebesgue measurable. Therefore $\underline{S}(f)$ and $\overline{S}(f)$ are well-defined and satisfy*

$$-\infty < \underline{S}(f) \leq \overline{S}(f) < +\infty.$$

For properties of these indices we refer to [Mer84] and [CF88].

The result we present next (cf. [BM03, p. 384, Proposition 3.9]) relates the Boyd indices of an admissible sequence with the corresponding indices for functions in $\mathbf{B}_\sigma$.

Proposition 1.2.20. *Let σ be an admissible sequence and $f \in \mathbf{B}_\sigma$. Then*

$$\underline{s}(\sigma) = \underline{S}(f) \qquad \textit{and} \qquad \overline{s}(\sigma) = \overline{S}(f).$$

1.3 Basic notation in measure theory

We collect some basic notation about measures on metric spaces. For more details we refer to [Mat95, pp. 7-13] and [Fal86, pp. 1-6].

Definition 1.3.1. *Let X be a metric space. A set function $\mu : \{A : A \subset X\} \to [0, \infty]$ is called a measure if*

$$\mu(\emptyset) = 0;$$

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i) \quad A \subset \bigcup_{i=1}^{\infty} A_i, \quad A_i \subset X.$$

Definition 1.3.2. *Let μ be a measure and $A \subset X$.*

(i) We say that A is μ-measurable if

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A) \quad \textit{for all} \quad E \subset X.$$

(ii) We say that A is a Borel set if it belongs to the smallest σ-algebra containing the open subsets of X.

(iii) The restriction of μ to A, denoted by $\mu|_A$ is given by

$$(\mu|_A)(B) = \mu(A \cap B) \quad \textit{where} \quad B \subset X.$$

Remark 1.3.3. *The collection of the μ-measurable sets is a σ-algebra.*

Definition 1.3.4. *(i) A measure μ is called finite if $\mu(X) < \infty$ and is called locally finite if every compact set of X has finite μ-measure.*

(ii) A measure μ is said to be a Borel measure if all Borel sets are μ-measurable.

(iii) A measure μ is called Borel regular if it is a Borel measure and if for any $A \subset X$ there is a Borel set $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.

(iv) A measure μ on X is called a Radon measure if it is a locally finite Borel measure and
$\mu(O) = \sup\{\mu(K) : K$ compact $K \subset O\}$ for all open sets $O \subset X$,
$\mu(A) = \inf\{\mu(G) : G$ open, $A \subset G\}$ for all sets $A \subset X$.

Definition 1.3.5. *Let μ be a Borel measure. The support of μ is given by*

$$\operatorname{supp} \mu = X \setminus \cup\{O : O \textit{ open}, \ \mu(O) = 0\}.$$

1.4 Hausdorff measures and Hausdorff dimensions

In this section we present the definitions of Hausdorff measures and Hausdorff dimensions. For more details we refer to [Mat95, Chapter 4], [Tri97, Chapter I, Section 2] and [Bri01, Chapter 1].

Definition 1.4.1. *Let $\mathbb{H}$ denote the class of all continuous monotone increasing functions $h : (0, \infty) \to (0, \infty)$ such that $h(0^+) = 0$. We refer to $\mathbb{H}$ as the set of all gauge functions.*

Definition 1.4.2. *Let $h \in \mathbb{H}$. Let $h^*(\emptyset) := 0$ and, for $A \subset \mathbb{R}^n$ with $\operatorname{diam}(A) \leq 1$, let $h^*(A) := h(\operatorname{diam}(A))$. The set function*

$$\mathcal{H}^h(A) := \lim_{\substack{\delta \to 0^+ \\ \delta \leq 1}} \mathcal{H}^h_\delta(A), \quad A \subset \mathbb{R}^n,$$

where

$$\mathcal{H}^h_\delta(A) := \inf\{\sum_{i=1}^{\infty} h^*(A_i) : A_i \textit{ open}, \operatorname{diam}(A_i) < \delta \textit{ and } \cup_{i=1}^{\infty} A_i \supset A\},$$

is called Hausdorff measure corresponding to the gauge function h.

Remark 1.4.3. *If $h(r) = r^s$, for some $s \geq 0$, we shall write $\mathcal{H}^{(s)}$ instead of $\mathcal{H}^h$.*

Definition 1.4.4. *Let $A \subset \mathbb{R}^n$. The Hausdorff dimension of A is given by*

$$\mathit{dim}_{\mathcal{H}}\, A := \inf\{s \geq 0 \,:\, \mathcal{H}^{(s)}(A) = 0\}.$$

Chapter 2

Besov spaces of generalised smoothness on Euclidean n-spaces

In this chapter we present a collection of results on Besov spaces of generalised smoothness on $\mathbb{R}^n$. Though in the present chapter we will consider only such spaces, in part of Chapter 5 we will deal with particular classes of these Besov spaces which coincide with Sobolev and Bessel-potential spaces. So let us recall the definition of the latter.

Definition 2.0.5. *Let $1 < p < \infty$.*

(i) Let $m \in \mathbb{N}$. The Sobolev space on $\mathbb{R}^n$, $W_p^m(\mathbb{R}^n)$, is given by

$$W_p^m(\mathbb{R}^n) := \{f \in L_p(\mathbb{R}^n) : \|f|W_p^m(\mathbb{R}^n)\| = \sum_{|\alpha|\leq m} \|D^\alpha f|L_p(\mathbb{R}^n)\| < \infty\}.$$

(ii) Let $s \in \mathbb{R}$. The Bessel-potential space on $\mathbb{R}^n$, $H_p^s(\mathbb{R}^n)$, is given by

$$H_p^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|\mathcal{F}^{-1}((1+|x|^2)^{s/2}\mathcal{F}f)|L_p(\mathbb{R}^n)\| < \infty\}.$$

If $p = 2$ we use the abbreviation

$$H_p^s(\mathbb{R}^n) = H^s(\mathbb{R}^n).$$

We start this chapter defining Besov spaces of generalised smoothness on $\mathbb{R}^n$ by a Fourier-analytical approach and collecting some well-known properties. Then we present characterisations of these spaces with quarkonial and smooth atomic decompositions. We prove a characterisation by differences and apply it to obtain a homogeneity property. Finally, applying the previous results we prove a characterisation with non-smooth atomic decompositions.

2.1 The Fourier-analytical approach and some properties

We fix some more notation.

Definition 2.1.1. *For N as in Convention 1.2.10 and Remark 1.2.11 we define the associated covering $\Omega^N := (\Omega_j^N)_{j\in\mathbb{N}_0}$ of $\mathbb{R}^n$ by*

$$\Omega_j^N = \{\xi \in \mathbb{R}^n : |\xi| \le N_{j+l_0}\}, \quad j = 0, 1, ...l_0 - 1,$$

and

$$\Omega_j^N = \{\xi \in \mathbb{R}^n : N_{j-l_0} \le |\xi| \le N_{j+l_0}\}, \quad j \ge l_0.$$

A system $\varphi^N = (\varphi_j^N)_{j\in\mathbb{N}_0}$ will be called a (generalised) partition of unity subordinated to Ω^N if:

(i)

$$\varphi_j^N \in C_0^\infty(\mathbb{R}^n) \quad \text{and} \quad \varphi_j^N(\xi) \ge 0 \quad \text{if} \quad \xi \in \mathbb{R}^n \quad \text{for any} \quad j \in \mathbb{N}_0;$$

(ii)

$$\operatorname{supp} \varphi_j^N \subset \Omega_j^N \quad \text{for any} \quad j \in \mathbb{N}_0;$$

(iii) for any $\alpha \in \mathbb{N}_0^n$ there exists a constant $c_\alpha > 0$ such that for any $j \in \mathbb{N}_0$

$$|D^\alpha \varphi_j^N(\xi)| \le c_\alpha (1 + |\xi|^2)^{-\frac{|\alpha|}{2}} \quad \text{for any} \quad \xi \in \mathbb{R}^n;$$

(iv) for all $\xi \in \mathbb{R}^n$

$$\sum_{j=0}^{\infty} \varphi_j^N(\xi) = 1.$$

By the Paley-Wiener-Schwartz theorem (cf. [Tri83, p. 13], for example), for any $f \in \mathcal{S}'(\mathbb{R}^n)$, $(\varphi_j^N \widehat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$. In particular, $(\varphi_j^N \widehat{f})^\vee$ makes sense pointwise. Moreover

$$f = \sum_{j=0}^{\infty} (\varphi_j^N \widehat{f})^\vee$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$.

Next we define Besov and Triebel-Lizorkin spaces of generalised smoothness on $\mathbb{R}^n$, according to [FL06]:

Definition 2.1.2. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence, $N = (N_j)_{j\in\mathbb{N}_0}$ and $\varphi^N = (\varphi_j^N)_{j\in\mathbb{N}_0}$ be as in Definition 2.1.1.*

(i) Let $0 < p, q \leq \infty$. The Besov space of generalised smoothness on $\mathbb{R}^n$ is given by

$$B^{\sigma,N}_{p,q}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^{\sigma,N}_{p,q}(\mathbb{R}^n)\| = \|(\sigma_j \varphi_j^N(D) f)_{j\in\mathbb{N}_0}|\ell_q(L_p)\| < \infty\}.$$

(ii) Let $0 < p < \infty$ and $0 < q \leq \infty$. The Triebel-Lizorkin space of generalised smoothness on $\mathbb{R}^n$ is given by

$$F^{\sigma,N}_{p,q}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f|F^{\sigma,N}_{p,q}(\mathbb{R}^n)\| = \|(\sigma_j \varphi_j^N(D) f)_{j\in\mathbb{N}_0}|L_p(\ell_q)\| < \infty\}.$$

Remark 2.1.3. *Let $A \in \{B, F\}$ and let $A^{\sigma,N}_{p,q}(\mathbb{R}^n)$ denote the spaces $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ if $A = B$ and the spaces $F^{\sigma,N}_{p,q}(\mathbb{R}^n)$ if $A = F$.*
The spaces $A^{\sigma,N}_{p,q}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$). In particular, using the fact that both ℓ_p and $L_p(\mathbb{R}^n)$ are η-Banach spaces, with $\eta := \min\{1, p\}$, one can check that the spaces $A^{\sigma,N}_{p,q}(\mathbb{R}^n)$ are t-Banach spaces for $t := \min\{1, p, q\}$.
The spaces are independent of the system $\varphi^N = (\varphi_j^N)_{j\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms). This Fourier analytic description of Besov spaces of generalised smoothness was given in [FL06]. In this work one can also find some information about the history of function spaces of generalised smoothness with several references.
If $p = q$ we abbreviate $A^{\sigma,N}_{p}(\mathbb{R}^n) = A^{\sigma,N}_{p,p}(\mathbb{R}^n)$.

If $N_j = 2^j, j \in \mathbb{N}_0$*, we have the Besov or the Triebel-Lizorkin spaces of generalised smoothness studied by Bricchi in [Bri01] and we write* $A^{\sigma}_{p,q}(\mathbb{R}^n)$*.*
If $N = (2^j)_{j\in\mathbb{N}_0}$ *and* $\sigma = (s)$ *for some* $s \in \mathbb{R}$*, the above spaces coincide with the usual Besov or Triebel-Lizorkin spaces usually denoted by* $A^{s}_{p,q}(\mathbb{R}^n)$ *and treated in detail by Triebel in [Tri83], [Tri92b] and [Tri01]. We will follow the notation referred above and denote these spaces by* $A^{(s)}_{p,q}(\mathbb{R}^n)$*.*
As it was mentioned in [FL06, p. 26], as in the classic case, $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $A^{\sigma,N}_{p,q}(\mathbb{R}^n)$*, for* $0 < p, q < \infty$*.*
It is well-known that Besov and Triebel-Lizorkin spaces include classic function spaces, namely:

(i) $F^{(0)}_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ *if* $1 < p < \infty$*;*

(ii) $F^{(m)}_{p,2}(\mathbb{R}^n) = W^m_p(\mathbb{R}^n)$ *if* $1 < p < \infty$ *and* $m \in \mathbb{N}$*;*

(iii) $F^{(s)}_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$ *if* $1 < p < \infty$ *and* $s \in \mathbb{R}$*;*

(iv) $B^{(s)}_{\infty,\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n)$ *if* $s > 0$*,*

where $\mathcal{C}^s(\mathbb{R}^n)$*, for* $s > 0$*, denotes the Hölder-Zygmund spaces. For the definition we refer to [Tri83, p. 36]*

In the next theorem (cf. [Bri01, p. 55, Proposition 2.2.14]) the dual spaces of the spaces $B^{\sigma}_{p,q}(\mathbb{R}^n)$ are determined.

Theorem 2.1.4. *Let* σ *be an admissible sequence,* $1 \le p < \infty$*,* $1 \le q < \infty$ *and* p' *and* q' *denote their conjugates. Then*

$$(B^{\sigma}_{p,q}(\mathbb{R}^n))' = B^{\sigma^{-1}}_{p',q'}(\mathbb{R}^n), \tag{2.1.1}$$

according with the notation introduced in (1.2.3)*.*

We present now some well-known embeddings that will be applied frequently in the whole work.

Proposition 2.1.5. *Let $0 < q \leq \infty$ and σ and τ be admissible sequences.*
(i) Let $0 < p, r \leq \infty$. Then

$$B^{\sigma}_{p,q}(\mathbb{R}^n) \subset B^{\tau}_{p,r}(\mathbb{R}^n) \quad \textit{if} \quad \sigma^{-1}\tau \in \ell_{(q_r)'}, \tag{2.1.2}$$

where $(q_r)' \in [1, \infty]$ is given by

$$\begin{cases} \frac{1}{(q_r)'} + \frac{1}{q} = \frac{1}{r} & , \quad \textit{if} \quad q > r \\ \\ \infty & , \quad \textit{if} \quad q \leq r. \end{cases}$$

In particular,

$$B^{\sigma}_{p,q}(\mathbb{R}^n) \subset B^{\tau}_{p,r}(\mathbb{R}^n) \quad \textit{if} \quad \overline{s}(\tau) < \underline{s}(\sigma). \tag{2.1.3}$$

(ii) Let $0 < p_0 \leq p_1 \leq \infty$. Then

$$B^{\sigma}_{p_0,q}(\mathbb{R}^n) \subset B^{\tau}_{p_1,q}(\mathbb{R}^n) \quad \textit{if} \quad \sigma^{-1}\tau(n)^{\frac{1}{p_0}-\frac{1}{p_1}} \in \ell_{\infty}. \tag{2.1.4}$$

(iii) Let $0 < p \leq \infty$ and suppose that $\underline{s}(\sigma) > n(\frac{1}{p} - 1)_+$. Then

$$B^{\sigma}_{p,q}(\mathbb{R}^n) \subset L_p(\mathbb{R}^n), \quad \textit{if} \quad p \geq 1, \tag{2.1.5}$$

and

$$B^{\sigma}_{p,q}(\mathbb{R}^n) \subset L_1(\mathbb{R}^n) \cap L_p(\mathbb{R}^n), \quad \textit{if} \quad 0 < p < 1. \tag{2.1.6}$$

Remark 2.1.6. *The embedddings in* (2.1.2) *and* (2.1.4)-(2.1.6) *were stated in [Bri01, p. 56]. The result in* (2.1.3) *follows from* (2.1.2) *and* (1.2.4).

It is useful for us to deal in the context of $\mathbb{R}^n$ with powers $2^{-\varepsilon j}$, $j \in \mathbb{N}_0$, besides the usual 2^{-j}, $j \in \mathbb{N}_0$. The following proposition clarifies how to switch from one case to the other. It is a consequence of a result proved by Caetano and Leopold (cf. [CL06], p. 432, Theorem 1). This Theorem is a standardisation result for Besov and Triebel-Lizorkin spaces of generalised smoothness, allowing the reduction, under suitable hypotheses, to corresponding spaces with the usual dyadic decomposition on the Fourier side. Hence, it states that, under some conditions, spaces $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ can be reduced to spaces $B^{\beta}_{p,q}(\mathbb{R}^n)$ and the construction of the sequence β is given.

Proposition 2.1.7. *Let* $0 < \varepsilon \leq 1$, $N_\varepsilon = (2^{\varepsilon j})_{j \in \mathbb{N}_0}$, σ *be an admissible sequence and* σ_ε *be as in* (1.2.6). *Then*

$$B^{\sigma}_{p,q}(\mathbb{R}^n) = B^{\sigma_\varepsilon, N_\varepsilon}_{p,q}(\mathbb{R}^n) \tag{2.1.7}$$

(equivalent quasi-norms).

Proof. By [CL06, p. 432, Theorem 1], the sequence β, defined by

$$\beta_j := \sigma_{\varepsilon,k(j)}, \quad \text{where} \quad k(j) := \min\{k \in \mathbb{N}_0 \,:\, 2^{j-1} \leq 2^{\varepsilon(k+[1/\varepsilon]+1)}\}, \quad j \geq 1, \tag{2.1.8}$$

and $\beta_0 := \sigma_{\varepsilon,k(1)}$, is an admissible sequence and

$$B^{\sigma_\varepsilon, N_\varepsilon}_{p,q}(\mathbb{R}^n) = B^{\beta}_{p,q}(\mathbb{R}^n).$$

Therefore, to prove (2.1.7) it is enough to prove that $\beta \sim \sigma$. As

$$\sigma_j \sim \Lambda(2^j) \quad \text{and} \quad \beta_j := \sigma_{\varepsilon,k(j)} = \Lambda(2^{\varepsilon k(j)}), \quad j \in \mathbb{N},$$

and by Remark 1.2.13, we just need to guarantee that there exist $c_0, c_1 > 0$ such that

$$c_0 2^{\varepsilon k(j)} \leq 2^j \leq c_1 2^{\varepsilon k(j)}, \quad j \in \mathbb{N}. \tag{2.1.9}$$

It follows immediately from (2.1.8) that

$$2^{\varepsilon(k(j)-1+[1/\varepsilon]+1)} \leq 2^{j-1} \leq 2^{\varepsilon(k(j)+[1/\varepsilon]+1)}, \quad j \in \mathbb{N},$$

which implies (2.1.9), concluding the proof. $\square$

Remark 2.1.8. *It follows from Proposition 2.1.7 that, under the same conditions, for all* $k \in \mathbb{N}_0$,

$$B^{T_{[\varepsilon k]}(\sigma)}_{p,q}(\mathbb{R}^n) = B^{T_k(\sigma_\varepsilon), N_\varepsilon}_{p,q}(\mathbb{R}^n) \tag{2.1.10}$$

(equivalent quasi-norms), where we are using the notation introduced in Proposition 1.2.15. Following the proof of [CL06, Theorem 1, p. 432] one can conclude that the equivalence constants involved in (2.1.10) *are independent of* k.

2.2 Characterisation by quarkonial and smooth atomic decompositions

In this section characterisations of Besov spaces with generalised smoothness with quarkonial and smooth atomic decompositions will be presented. First we describe the concepts and notation involved.

Definition 2.2.1. *Let $0 < \varepsilon \leq 1$. We say that*

$$\{y^{j,l} : l \in \mathbb{Z}^n\} \subset \mathbb{R}^n,$$

with $j \in \mathbb{N}_0$, is a $2^{-\varepsilon j}$-approximate lattice if there exist positive numbers $c_{\varepsilon,1}$ and $c_{\varepsilon,2}$ such that

$$|y^{j,l_1} - y^{j,l_2}| \geq c_{\varepsilon,1} 2^{-\varepsilon j}, \quad j \in \mathbb{N}_0, \quad l_1 \neq l_2, \tag{2.2.1}$$

and

$$\mathbb{R}^n = \bigcup_{l \in \mathbb{Z}^n} B(y^{j,l}, c_{\varepsilon,2} 2^{-\varepsilon j}). \tag{2.2.2}$$

We say that

$$\{\theta^{j,l} : l \in \mathbb{Z}^n\},$$

$j \in \mathbb{N}_0$, is a resolution of unity subordinated to $\{y^{j,l}\}_l$ if $\theta^{j,l}$ are, for all $l \in \mathbb{Z}^n$, non-negative C^∞ functions in $\mathbb{R}^n$ with

$$\operatorname{supp} \theta^{j,l} \subset B(y^{j,l}, d2^{-\varepsilon j}), \tag{2.2.3}$$

for a fixed $d > c_{\varepsilon,2}$ and such that

$$|D^\alpha \theta^{j,l}(x)| \leq c_\alpha 2^{\varepsilon j |\alpha|}, \quad \textit{for all} \quad x \in \mathbb{R}^n, \quad \alpha \in {\mathbb{N}_0}^n,$$

and

$$\sum_{l \in \mathbb{Z}^n} \theta^{j,l}(x) = 1, \quad x \in \mathbb{R}^n.$$

Assumption 2.2.2. *In what follows, in all results involving approximate lattices and subordinated resolutions of unity, we assume that they are fixed.*

Example 2.2.3. *Let $0 < \varepsilon \leq 1$. Let ω be a mother function, i.e., a non-negative smooth function such that*

$$\operatorname{supp} \omega \subset \{x \in \mathbb{R}^n : |x| < 2^\rho\}, \quad \textit{for some} \quad \rho > 0$$

and

$$\sum_{m \in \mathbb{Z}^n} \omega(x - m) = 1, \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

For all $j \in \mathbb{N}_0$,

$$\{2^{-\varepsilon j} l : l \in \mathbb{Z}^n\}$$

is a $2^{-\varepsilon j}$-approximate lattice and

$$\{\omega(2^{\varepsilon j} \cdot - l) : l \in \mathbb{Z}^n\}$$

is a subordinated resolution of unity.

When the particular lattices of the above example are considered, they are usually related to the cubes we describe next.

Definition 2.2.4. *Let $0 < \varepsilon \leq 1$, $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$, $l = (l_1, ..., l_n)$. We denote by $Q_{\varepsilon j, l}$ the half-open cube in $\mathbb{R}^n$ with center at $2^{-\varepsilon j} l$, sides parallel to the coordinate axes and side length $2^{-\varepsilon j}$,*

$$Q_{\varepsilon j, l} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : \frac{-2^{-\varepsilon j}}{2} < 2^{-\varepsilon j} l_i - x_i \leq \frac{2^{-\varepsilon j}}{2},\ i = 1, ..., n\}.$$

The following functions are also usually considered in this context.

Definition 2.2.5. *Let $0 < \varepsilon \leq 1$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $0 < p \leq \infty$. We denote by $\chi^{(p)}_{\varepsilon j, m}$ the p-normalised characteristic function of the cube $Q_{\varepsilon j, m}$, i.e.,*

$$\chi^{(p)}_{\varepsilon j, m} = 2^{\frac{\varepsilon j n}{p}} \ \textit{if} \ x \in Q_{\varepsilon j, m} \quad \textit{and} \quad \chi^{(p)}_{\varepsilon j, m} = 0 \ \textit{if} \ x \notin Q_{\varepsilon j, m}.$$

In the characterisations of elements of Besov spaces with quarkonial and atomic decompositions, the elements of the space are, roughly speaking, given as the limit of a sum of the product of quarks or atoms by coefficients. These coefficients shall be part of a sequence in convenient spaces. Next we present spaces of sequences that will be considered.

Definition 2.2.6. *Let $0 < p, q \leq \infty$, $0 < \varepsilon \leq 1$ and*

$$\lambda = \{\lambda_{j,l} \in \mathbb{C} : j \in \mathbb{N}_0, l \in \mathbb{Z}^n\}.$$

Then

$$\|\lambda|b_{p,q}\| := \Bigg(\sum_{j=0}^{\infty}\Big\|\sum_{l\in\mathbb{Z}^n}\lambda_{j,l}\chi^{(p)}_{\varepsilon j,l}|L_p(\mathbb{R}^n)\Big\|^q\Bigg)^{\frac{1}{q}} = \Bigg(\sum_{j=0}^{\infty}\Big(\sum_{l\in\mathbb{Z}^n}|\lambda_{j,l}|^p\Big)^{\frac{q}{p}}\Bigg)^{\frac{1}{q}}$$

(with the usual modification if $p = \infty$ or $q = \infty$) and

$$b_{p,q} := \Big\{\lambda : \|\lambda|b_{p,q}\| < \infty\Big\}.$$

If $p = q$ we use the abbreviation $b_p = b_{p,p}$.

Definition 2.2.7. *Let $0 < \varepsilon \leq 1$, σ be an admissible sequence and $0 < p \leq \infty$. Consider, for all $j \in \mathbb{N}_0$, a fixed $2^{-\varepsilon j}$-approximate lattice and a subordinated resolution of the unity as in Definition 2.2.1. Then the function defined by*

$$(\beta\text{-}qu)_{j,l}(x) := \sigma_j^{-1}2^{j\varepsilon(\frac{n}{p}+|\beta|)}(x - y^{j,l})^\beta\theta^{j,l}(x), \quad x \in \mathbb{R}^n,\ \beta \in \mathbb{N}_0^n,\ l \in \mathbb{Z}^n,\ j \in \mathbb{N}_0,$$

is called a β-(σ, p)-ε-quark (located at $B(y^{j,l}, c_{\varepsilon,2}2^{-\varepsilon j})$).

Theorem 2.2.8. *Let $0 < p, q \leq \infty$, $k \in \mathbb{N}_0$, $0 < \varepsilon \leq 1$, $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+$$

Fix $\rho > a$ where a is such that $d = 2^{\varepsilon a}$, for d as in (2.2.3).
Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f(x) = \sum_{\beta\in\mathbb{N}_0^n}\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\lambda^\beta_{j,l}(\beta\text{-}qu)_{j,l}(x), \tag{2.2.4}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$, where $(\beta\text{-}qu)_{j,l}$ are β-$(T_k(\sigma_\varepsilon), p)$-$\varepsilon$-quarks according to Definition 2.2.7, $\lambda^\beta \in b_{p,q}$, for all $\beta \in \mathbb{N}_0^n$, and

$$\|\lambda|b_{p,q}\|_\rho := \sup_{\beta\in\mathbb{N}_0^n} 2^{\rho|\beta|\varepsilon}\|\lambda^\beta|b_{p,q}\| < \infty. \tag{2.2.5}$$

Furthermore,

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \sim \inf \|\lambda|b_{p,q}\|_\rho \tag{2.2.6}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (2.2.4)*. The equivalence constants in* (2.2.6) *are independent of* k*.*

Remark 2.2.9. *In [Tri01, Theorem 2.9, pp. 15-16] a characterisation of classical Besov spaces*

$$B^{(s)}_{p,q}(\mathbb{R}^n), \quad 0 < p, q \leq \infty, \quad s > n\Big(\frac{1}{p} - 1\Big)_+,$$

with quarkonial decompositions was given. In this case quarks were considered located in dilations of the cubes $Q_{j,m}$ *and were defined through dilations and translations of one fixed smooth function with compact support, usually called mother function.*
In [Mou01b, Theorem 1.23, pp. 35-36] this result was extended to Besov spaces spaces of generalised smoothness

$$B^{(s,\psi)}_{p,q}(\mathbb{R}^n), \quad 0 < p, q \leq \infty, \quad s > n\Big(\frac{1}{p} - 1\Big)_+,$$

where (s, ψ) *is as in Example 1.2.2(ii), and in [Bri01, Theorem 2.3.19, p. 76] it was extended to the spaces*

$$B^{\sigma}_{p,q}(\mathbb{R}^n), \quad 0 < p, q \leq \infty, \quad \underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+,$$

where σ *is an admissible sequence.*
In [KZ06, Theorem 10] a characterisation was presented with decompositions with quarks for Besov spaces of generalised smoothness

$$B^{\sigma,N}_{p,q}(\mathbb{R}^n), \quad 1 < p, q < \infty, \quad \sigma \textit{ increasing admissible sequence.}$$

In this case more general quarks were considered, located in approximate lattices and defined from subordinated resolutions of unity of the same kind of the ones described in Definition 2.2.1.
In Theorem 2.2.8 we consider general quarks located in approximate lattices, as in [KZ06], but we extend the result to $0 < p, q \leq \infty$ *as in [Tri01, Bri01]. We consider a smaller class*

of sequences N *than in [KZ06]:* $N = (2^{\varepsilon j})_j$*, for some* $0 < \varepsilon \leq 1$*. Moreover we consider sequences* $T_k(\sigma_\varepsilon)$ *(according to the notation given in* (1.2.11)*), instead of the usual* σ*, and we guarantee the independence of* k *in the constants in* (2.2.6)*. We present parts of the proof of this result at the end of this section, combined with the proof of a characterisation with smooth atomic decompositions.*

Now we define the atoms that will be considered. This definition is an adaptation, for our purposes, of the one considered in [FL06, 4.4].

Definition 2.2.10. *Let* $K \in \mathbb{N}_0$*,* $0 < \varepsilon \leq 1$ *and* σ *an admissible sequence. Consider, for all* $j \in \mathbb{N}_0$*, a fixed* $2^{-\varepsilon j}$*-approximate lattice as in Definition 2.2.1. Let* $d > c_{\varepsilon,2}$*, where* $c_{\varepsilon,2}$ *is as in* (2.2.2)*.*
(i) A function $a \in \mathcal{C}^K(\mathbb{R}^n)$ *is called a* d*-*σ*-*1_K*-*ε*-atom if*

$$\operatorname{supp} a \subset B(y^{0,l}, d), \qquad \textit{for some} \quad l \in \mathbb{Z}^n,$$

and

$$\sup_{x\in\mathbb{R}^n} |D^\alpha a(x)| \leq \sigma_0^{-1} \quad \textit{for} \quad |\alpha| \leq K.$$

(ii) Additionally, let $0 < p \leq \infty$ *and* $L \in \mathbb{N}_0 \cup \{-1\}$*. A function* $a \in \mathcal{C}^K(\mathbb{R}^n)$ *is called a* d*-*$(\sigma, p)_{K,L}$*-*ε*-atom if for some* $j \in \mathbb{N}$

$$\operatorname{supp} a \subset B(y^{j,l}, d2^{-\varepsilon j}) \quad \textit{for some} \quad l \in \mathbb{Z}^n;$$

$$\sup_{x\in\mathbb{R}^n} |D^\alpha a(x)| \leq \sigma_j^{-1}\, 2^{\frac{\varepsilon j n}{p}}\, 2^{\varepsilon|\alpha| j} \quad \textit{for} \quad |\alpha| \leq K.$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)dx = 0 \quad \textit{if} \quad |\beta| \leq L. \tag{2.2.7}$$

Remark 2.2.11. *If* $L = -1$ *then no moment condition* (2.2.7) *is required. In this case we omit the subscript "L" and we simply speak of* d*-*$(\sigma, p)_K$*-*ε*-atoms. We say for an atom as above that it is located at* $B(y^{j,l}, d2^{-\varepsilon j})$ *and we shall denote it by* $a^{j,l}$*.*

Finally we present the theorem which gives a characterisation of some spaces $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ with decompositions with conveniently chosen smooth atoms.

Theorem 2.2.12. *Let $0 < p, q \le \infty$, $k \in \mathbb{N}_0$, $0 < \varepsilon \le 1$, $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+$$

Fix $K \in \mathbb{N}_0$ with $K > \overline{s}(\sigma)$ and $d > c_{\varepsilon,2}$, for $c_{\varepsilon,2}$ as in (2.2.2). Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f(x) = \sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n} \nu_{j,l} a^{j,l}(x), \tag{2.2.8}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$, where $a^{j,l}$ are d-$T_k(\sigma_\varepsilon)$-1_K-ε-atoms ($j = 0$) or d-$(T_k(\sigma_\varepsilon), p)_K$-$\varepsilon$-atoms ($j \in \mathbb{N}$) according to Definition 2.2.10 and Remark 2.2.11, and $\nu \in b_{p,q}$. Furthermore,

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \sim \inf \|\nu|b_{p,q}\| \tag{2.2.9}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (2.2.8). The equivalence constants in (2.2.9) are independent of k.

Remark 2.2.13. *For a historical report on atomic decompositions for the elements of function spaces we refer to [Tri92b, Section 1.9.1]. For results on (smooth) atomic decompositions in the classical Besov spaces, $B^{(s)}_{p,q}(\mathbb{R}^n)$, we refer to [FJ85], [FJ90], [FJW91] and [Tri97]. In these works atoms are defined as being located at dilations of cubes $Q_{j,l}$, $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$. In [FL06] this result was extended to spaces $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$. In Theorem 2.2.12 we consider atoms located in balls with center in the elements of approximate lattices, as in Definition 2.2.1. Again, we consider a smaller class of sequences N than in [FL06]: $N = (2^{\varepsilon j})_j$, for some $0 < \varepsilon \le 1$. And, once more, it is essential for our purposes to consider, instead of the usual admissible sequence σ, sequences $T_k(\sigma_\varepsilon)$ (we refer to (1.2.11)) and to guarantee the independence of k in the constants in (2.2.9).*

Proof. We present part of the proofs of Theorems 2.2.8 and 2.2.12. The proof was obtained in circle in the following way: The "if" part of Theorem 2.2.12 can be proved following the proof in [FL06].

The "only if" part of Theorem 2.2.8 follows by adapting the corresponding proofs in [Tri01, Mou01b, Bri01, KZ06]. We prove now that if a tempered distribution can be represented with quarkonial decompositions, then it can be represented with atomic decompositions. So let $f \in \mathcal{S}'(\mathbb{R}^n)$ be as in (2.2.4), with (2.2.5). It was proved in previous works that (2.2.5) implies that (2.2.4) converges unconditionally in $L_p(\mathbb{R}^n)$ and, if $0 < p \leq 1$, also in $L_1(\mathbb{R}^n)$. We refer to [Tri01, p. 14] and [Bri01, pp. 77-78] and [Mou01a, pp. 51-53].

We remark that, by (2.2.5),

$$\|\lambda^\beta|b_{p,q}\| \leq \|\lambda|b_{p,q}\|_\rho \cdot 2^{-\varepsilon\rho|\beta|}, \quad \beta \in \mathbb{N}_0^n.$$

Let $K \in \mathbb{N}_0$ be such that $K > \overline{s}(\sigma)$. For all $b > 0$ there is a positive number c_b such that

$$|\beta|^K \leq c_b 2^{\varepsilon b|\beta|}.$$

We fix $b \in (0, \rho - a)$, where a is such that $d = 2^{\varepsilon a}$, for d as in (2.2.3). If $(\beta\text{-}qu)_{j,l}$ is a β-$(T_k(\sigma_\varepsilon), p)$-$\varepsilon$-quark, then its support is contained in the the ball $B(y^{j,l}, d2^{-\varepsilon j})$ and it can be proved that for all $\alpha \in \mathbb{N}_0^n$, with $|\alpha| \leq K$,

$$|D^\alpha(\beta\text{-}qu)_{j,l}(x)| \leq c_K\, c_b\, \sigma_{\varepsilon,j+k}^{-1} 2^{\frac{\varepsilon jn}{p}} 2^{\varepsilon j|\alpha|} 2^{\varepsilon|\beta|(a+b)}, \quad x \in \mathbb{R}^n.$$

We define, for $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$, $T \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$, with $|\alpha| \leq K$,

$$f_{\alpha,T}^{j,l}(x) := \sum_{t=0}^{T} \sum_{|\beta|=t} \lambda_{j,l}^\beta D^\alpha(\beta\text{-}qu)_{j,l}(x). \tag{2.2.10}$$

We remark that

$$f_{\alpha,T}^{j,l} = D^\alpha f_{0,T}^{j,l}. \tag{2.2.11}$$

Then

$$|f_{\alpha,T}^{j,l}(x)| \leq c_K c_b \sigma_{\varepsilon,j+k}^{-1} 2^{\frac{\varepsilon jn}{p}} 2^{\varepsilon j|\alpha|} \sum_{t=0}^{T} \sum_{|\beta|=t} |\lambda_{j,l}^\beta| 2^{\varepsilon|\beta|(a+b)} \tag{2.2.12}$$

$$\leq c_K c_b \sigma_{\varepsilon,j+k}^{-1} 2^{\frac{\varepsilon jn}{p}} 2^{\varepsilon j|\alpha|} \|\lambda|b_{p,q}\|_\rho \sum_{t=0}^{T} 2^{-\varepsilon\rho t} 2^{\varepsilon t(a+b)} (t+1)^n \tag{2.2.13}$$

As $\rho > a + b$ then, for all $x \in \mathbb{R}^n$, $(f^{j,l}_{\alpha,T}(x))_{T\in\mathbb{N}_0}$ converges to, say, $f^{j,l}_{\alpha}(x)$. Moreover, for all $x \in \mathbb{R}^n$,

$$|f^{j,l}_{\alpha}(x)| \leq c_{a,b,\rho,K,\varepsilon}\sigma^{-1}_{\varepsilon,j+k}2^{\frac{\varepsilon jn}{p}}2^{\varepsilon j|\alpha|}\|\lambda|b_{p,q}\|_{\rho}.$$

For all j, l, α, the sequence of functions $(f^{j,l}_{\alpha,T})_{T\in\mathbb{N}_0}$ converges uniformly to $f^{j,l}_{\alpha}$. So (cf. [Lan93, Theorem 9.1, p. 356],

$$f^{j,l}_{\alpha} = D^{\alpha}f^{j,l}_{0}.$$

We fix $r \in (a + b, \rho)$. For $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, let

$$\nu_{j,l} := \sum_{\beta\in\mathbb{N}^n_0}|\lambda^{\beta}_{j,l}|2^{\varepsilon r|\beta|} \tag{2.2.14}$$

and

$$a^{j,l}(x) := \frac{f^{j,l}_0(x)}{\nu_{j,l}} \quad \text{if} \quad \nu_{j,l} \neq 0 \quad \text{and} \quad a^{j,l}(x) := 0 \quad \text{otherwise.} \tag{2.2.15}$$

Then it follows from (2.2.4) (unconditionally convergence) that

$$f = \sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\nu_{j,l}a^{j,l}.$$

From (2.2.10)-(2.2.12), (2.2.14) and (2.2.15), $a^{j,l}$ are, up to constants, d-$T_k(\sigma_\varepsilon)$-1_K-ε-atoms $(j = 0)$ or d-$(T_k(\sigma_\varepsilon), p)_K$-$\varepsilon$-atoms $(j \in \mathbb{N})$ located at $B(y^{j,l}, c_{\varepsilon,2}2^{-\varepsilon j})$. Let us prove that

$$\|\nu|b_{p,q}\| \leq c\|\lambda|b_{p,q}\|_{\rho}. \tag{2.2.16}$$

We present the proof for $0 < p, q < \infty$. The cases $p = \infty$ or $q = \infty$ are proved with the usual modifications. Let $\eta \in (0, (\rho - r)/2)$. For all $j \in \mathbb{N}_0$,

$$\begin{aligned}
\Big\|\sum_{l\in\mathbb{Z}^n}\nu_{j,l}\chi^{(p)}_{\varepsilon j,l}|L_p(\mathbb{R}^n)\Big\|^p &= \int_{\mathbb{R}^n}\Big(\sum_{\beta\in\mathbb{N}^n_0}\sum_{l\in\mathbb{Z}^n}|\lambda^{\beta}_{j,l}|2^{\varepsilon r|\beta|}2^{\frac{\varepsilon jn}{p}}\chi_{\varepsilon j,l}(x)\Big)^p dx \\
&\leq c\sum_{\beta\in\mathbb{N}^n_0}2^{(\eta+r)\varepsilon|\beta|p}2^{\varepsilon jn}\int_{\mathbb{R}^n}\Big(\sum_{l\in\mathbb{Z}^n}|\lambda^{\beta}_{j,l}|\chi_{\varepsilon j,l}(x)\Big)^p dx \quad (2.2.17) \\
&\leq c'\sum_{\beta\in\mathbb{N}^n_0}2^{(\eta+r)\varepsilon|\beta|p}\sum_{l\in\mathbb{Z}^n}|\lambda^{\beta}_{j,l}|^p
\end{aligned}$$

where (2.2.17) was obtained, for $1 < p < \infty$, applying Hölder inequality (if $0 < p \leq 1$ is immediate) and Beppo-Levi Theorem.

Let $\delta \in (0, (\rho - r)/2)$. Then

$$\begin{aligned}\sum_{j=0}^{\infty} \Big\| \sum_{l\in\mathbb{Z}^n} \nu_{j,l}\chi^{(p)}_{\varepsilon j,l} | L_p(\mathbb{R}^n)\Big\|^q &\leq c' \sum_{j=0}^{\infty} \Big(\sum_{\beta\in\mathbb{N}_0^n} 2^{(\eta+r)\varepsilon|\beta|p} \sum_{l\in\mathbb{Z}^n} |\lambda^{\beta}_{j,l}|^p \Big)^{q/p} \\ &\leq c'' \sum_{j=0}^{\infty} \sum_{\beta\in\mathbb{N}_0^n} 2^{(\eta+r+\delta)\varepsilon|\beta|q} \Big(\sum_{l\in\mathbb{Z}^n} |\lambda^{\beta}_{j,l}|^p \Big)^{q/p} \qquad (2.2.18)\end{aligned}$$

where (2.2.18) follows, if $1 < q/p < \infty$, from Hölder inequality (if $0 < q/p \leq 1$ is immediate). Therefore

$$\|\nu | b_{p,q}\|^q \lesssim \|\lambda | b_{p,q}\|^q_\rho \sum_{\beta\in\mathbb{N}_0^n} 2^{(\eta+r+\delta-\rho)\varepsilon|\beta|q} \lesssim \|\lambda | b_{p,q}\|^q_\rho,$$

proving (2.2.16).

□

2.3 Characterisation by approximation

In this section we collect some notation related to the characterisation of Besov spaces by approximation.

Definition 2.3.1. *Let $0 < \varepsilon \leq 1$, $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$ and $0 < p \leq \infty$. We define $\mathcal{U}_p^{N_\varepsilon}$ by*

$$\mathcal{U}_p^{N_\varepsilon} := \{a = (a_j)_{j\in\mathbb{N}_0} \ : \ a_j \in \mathcal{S}'(\mathbb{R}^n) \cap L_p(\mathbb{R}^n), \ \operatorname{supp} \widehat{a_j} \subset \{y : |y| \leq 2^{\varepsilon(j+l_0)}\}, \ j \in \mathbb{N}_0\}.$$

In the next theorem we present a characterisation by approximation of Besov spaces of generalised smoothness on $\mathbb{R}^n$. It is a modified version of [Mou07, Theorem 3.1].

Theorem 2.3.2. *Let $0 < \varepsilon \leq 1$ and $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$. Let $0 < p, q \leq \infty$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+.$$

Fix $k \in \mathbb{N}_0$. Then $f \in B^{T_k(\sigma_\varepsilon), N_\varepsilon}_{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and there exists $a \in \mathcal{U}_p^{N_\varepsilon}$ such that $f = \lim_{j\to\infty} a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|f | B^{T_k(\sigma_\varepsilon), N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_a := \sigma_{\varepsilon,k}\|a_0 | L_p(\mathbb{R}^n)\| + \|(\sigma_{\varepsilon,j+k}(f - a_j))_{j\in\mathbb{N}_0} | \ell_q(L_p)\| < \infty. \qquad (2.3.1)$$

Moreover

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_X := \inf \|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_a, \tag{2.3.2}$$

where the infimum is taken over all $a \in \mathcal{U}^{N_\varepsilon}_p$ *such that* $f = \lim_{j\to\infty} a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and* (2.3.1) *is satisfied, is an equivalent quasi-norm for* $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ *with equivalent constants independent of* k*.*

Remark 2.3.3. *In [Tri83] a characterisation by approximation for the classic Besov spaces was given. In [Mou07], it was extended to Besov spaces of generalised smoothness,* $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$*, which include the spaces* $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$*. In [Mou07, Theorem 3.1] it was proved that*

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_X := \inf \|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_a, \tag{2.3.3}$$

where the infimum is taken over all $a \in \mathcal{U}^{N_\varepsilon}_p$ *such that* $f = \lim_{j\to\infty} a_j$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and*

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_a := \|a_0|L_p(\mathbb{R}^n)\| + \|(\sigma_{\varepsilon,j+k}(f-a_j))_{j\in\mathbb{N}_0}|\ell_q(L_p)\|, \tag{2.3.4}$$

is an equivalent quasi-norm for the elements of $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$*. For our purposes it was convenient to obtain a similar characterisation guaranteeing that the equivalence constants involved are independent of* k*. Following the proof of [Mou07, Theorem 3.1] and adapting it one can prove that replacing* (2.3.3)-(2.3.4) *by* (2.3.1)-(2.3.2) *the independence of* k *is guaranteed.*

2.4 Characterisation by differences and a homogeneity property

In this section we present a characterisation by differences for Besov spaces of generalised smoothness and we apply it to prove a homogeneity property for these spaces. First, let us recall the definition of differences of functions. If f is an arbitrary complex-valued function on $\mathbb{R}^n$, $u \in \mathbb{R}^n$ and $M \in \mathbb{N}$, then

$$(\Delta^M_u f)(x) := \sum_{j=0}^{M} \binom{M}{j} (-1)^{M-j} f(x+ju), \quad x \in \mathbb{R}^n, \tag{2.4.1}$$

where $\binom{M}{j}$ are the binomial coefficients. The differences of functions can also be defined iteratively via

$$(\Delta_u^1 f)(x) = f(x+u) - f(x) \quad \text{and} \quad (\Delta_u^{k+1} f)(x) = \Delta_u^1(\Delta_u^k f)(x), \quad k \in \mathbb{N}.$$

Furthermore, the *k-th modulus of smoothness* of a function $f \in L_p(\mathbb{R}^n)$, $0 < p \leq \infty$, $k \in \mathbb{N}$, is defined by

$$\omega_k(f,t)_p := \sup_{|u| \leq t} \|\Delta_u^k f | L_p(\mathbb{R}^n)\|, \quad t > 0.$$

We need to fix some more notation. For $f \in L_p(\mathbb{R}^n)$ and $b > 0$ we define

$$E_p(b,f) := \inf \|f - g | L_p(\mathbb{R}^n)\| \tag{2.4.2}$$

where the infimum is taken over all $g \in \mathcal{S}'(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{g} \subset \{y : |y| \leq b\}$. According to [Tri83, Proposition 2.5.12, p. 110], given $M \in \mathbb{N}$ there is $c > 0$ such that

$$E_p(b,f) \leq c \sup_{|u| \leq b^{-1}} \|\Delta_u^M f | L_p(\mathbb{R}^n)\| \tag{2.4.3}$$

for all $b \geq 1$ and $f \in L_p(\mathbb{R}^n)$.

In [KL87] equivalent norms involving differences for some Besov spaces of generalised smoothness were presented. In [Mou07], Moura presented a characterisation by differences of Besov spaces with generalised smoothness. In the next theorem we present equivalent quasi-norms more convenient for what will be done in the next sections.

Theorem 2.4.1. *Let $0 < \varepsilon \leq 1$, $N_\varepsilon = (2^{\varepsilon j})_{j \in \mathbb{N}_0}$, $0 < p, q \leq \infty$, σ be an admissible sequence and $\Lambda \in \mathcal{A}_\sigma$. Consider*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+ \quad \textit{and} \quad \overline{s}(\sigma) < M \in \mathbb{N}. \tag{2.4.4}$$

For any $b \in (0, +\infty)$, $B_{p,q}^{T_k(\sigma_\varepsilon), N_\varepsilon}(\mathbb{R}^n)$ is the collection of all $f \in L_{\overline{p}}(\mathbb{R}^n)$, with $\overline{p} = \max\{1, p\}$, such that

$$\begin{aligned} \|f | B_{p,q}^{T_k(\sigma_\varepsilon), N_\varepsilon}(\mathbb{R}^n)\|_{b,\Lambda,M}^* \quad &:= \quad \sigma_{\varepsilon,k} \|f | L_p(\mathbb{R}^n)\| \\ &+ \quad \Big(\int_{|u| \leq b} \big(\Lambda(2^{\varepsilon k}|u|^{-1}) \omega_M(f, |u|)_p\big)^q \frac{du}{|u|^n} \Big)^{\frac{1}{q}} \end{aligned} \tag{2.4.5}$$

is finite or, equivalently, such that

$$\begin{aligned} \|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_{b,\Lambda,M} \quad &:= \quad \sigma_{\varepsilon,k}\|f|L_p(\mathbb{R}^n)\| \\ &+ \quad \left(\int_{|u|\le b}\left(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta^M_u f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n}\right)^{\frac{1}{q}} \end{aligned} \tag{2.4.6}$$

is finite (with the usual modification if $q=\infty$).
Moreover, $\|\cdot|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{b,M}$ and $\|\cdot|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_{b,M}$ are equivalent quasi-norms for $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ and the related equivalence constants are independent of k.

Remark 2.4.2. *Consider the conditions of Theorem 2.4.1 and fix $b\in(0,\infty)$ and $\Lambda\in\mathcal{A}_\sigma$. It can be proved that for all $f\in L_{\overline{p}}(\mathbb{R}^n)$, $\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{b,\Lambda,M}$ is finite if, and only if, $\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{1,\Lambda,M}$ is finite. Moreover*

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{b,\Lambda,M}\sim\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{1,\Lambda,M}, \tag{2.4.7}$$

where the equivalence constants are independent of f and k.
Let $\tau\in\mathcal{A}_\sigma$. Then it can be easily verified that

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{1,\Lambda,M}\sim\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{1,\tau,M}, \tag{2.4.8}$$

where the equivalence constants are independent of f and k. From (2.4.7)-(2.4.8) *follows that, given $b,b'\in(0,\infty)$ and $\Lambda,\tau\in\mathcal{A}_\sigma$,*

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{b,\Lambda,M}\sim\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{b',\tau,M}, \tag{2.4.9}$$

with equivalence constants independent of f and k. Analogously it can be proved that (2.4.9) *remains valid if we replace $\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{b,\Lambda,M}$ and $\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_{b',\tau,M}$ by $\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_{b,\Lambda,M}$ and $\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_{b',\tau,M}$, respectively. So in both cases different choices of b and Λ originate equivalent quasi-norms and this is the reason why, unless we consider convenient to distinguish them properly, we omit the subscripts b and Λ in our notation in what follows.*

Remark 2.4.3. *According to [Mou07], in the conditions considered,*

$$\|f|L_p(\mathbb{R}^n)\|+\left(\int_{|u|\le b}\left(\Lambda(2^{\varepsilon k}|u|^{-1})\omega_M(f,|u|)_p\right)^q\frac{du}{|u|^n}\right)^{\frac{1}{q}} \tag{2.4.10}$$

is an equivalent quasi-norm for the elements of $B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$*. In Moura's work there was no interest in taking sequences* $T_k(\sigma_\varepsilon)$ *and guaranteing the independence of* k*. Following the proof in [Mou07] adapted to our purposes, we concluded that one has to modify* (2.4.10) *by* (2.4.5) *to get equivalent quasi-norms where the related equivalence constants can be chosen independently of* k*.*

Moreover we also prove that there is no need to assume that the function is in the space in order to prove that the expressions in (2.4.5) *and* (2.4.6) *give equivalent quasi-norms. Actually we prove that, whenever one of them is finite, the tempered distribution must belong to the space.*

Proof. Step 1. If $f \in B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ then analogously to what was done in [Mou07] one proves that $f \in L_{\overline{p}}(\mathbb{R}^n)$ and that there is a positive number c, independent of k, such that

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_M \leq c\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|.$$

In this step we fix $f \in L_{\overline{p}}(\mathbb{R}^n)$ such that $\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_M$ is finite and we prove that

$$f \in B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n) \quad \text{and} \quad \|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \leq c'\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_M, \tag{2.4.11}$$

where c' is a positive number independent of f and k. By Remark 2.4.2 we may assume that $b = 1$ in (2.4.5).

Let

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_E := \sigma_{\varepsilon,k}\|f|L_p(\mathbb{R}^n)\| + \Big(\sum_{j=0}^{\infty} \sigma^q_{\varepsilon,j+k} E_p(2^{\varepsilon(j+l_0)}, f)^q\Big)^{1/q}, \tag{2.4.12}$$

where we are using the notation introduced in (2.4.2) and where l_0 satisfies (1.2.11) for the sequence $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$. For example, we may consider $l_0 = [1/\varepsilon] + 1$. It follows from (2.4.3) and (2.4.5) that

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_E \lesssim \|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^*_M. \tag{2.4.13}$$

Thus to prove (2.4.11) it is enough to prove that $f \in B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ and

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_X \leq c'\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_E, \tag{2.4.14}$$

with c' independent of k. Then (2.4.11) follows immediately by Theorem 2.3.2, (2.4.13) and (2.4.14). Let $\delta > 0$. By (2.4.13), the expression in (2.4.12) is finite. Hence for all $j \in \mathbb{N}_0$ there is $g_j \in \mathcal{S}'(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{g_j} \subset \{y : |y| \leq 2^{\varepsilon(j+l_0)}\}$ and

$$\|f - g_j|L_p(\mathbb{R}^n)\| \leq E_p(2^{\varepsilon(j+l_0)}, f) + \delta\sigma^{-1}_{\varepsilon,2j+k}.$$

Therefore, considering $\delta' \in (0, \underline{s}(\sigma))$,

$$\sum_{j=0}^{\infty} \sigma^q_{\varepsilon,j+k}\|f - g_j|L_p(\mathbb{R}^n)\|^q \leq c\sum_{j=0}^{\infty} \sigma^q_{\varepsilon,j+k}E_p(2^{\varepsilon(j+l_0)}, f)^q + c\delta^q\sum_{j=0}^{\infty} 2^{-\varepsilon jq(\underline{s}(\sigma)-\delta')} < \infty. \tag{2.4.15}$$

It follows from (2.4.15) that

$$\lim_{j\to\infty} g_j = f \quad \text{in} \quad L_p(\mathbb{R}^n). \tag{2.4.16}$$

If $p \geq 1$ then by (2.4.16), $(g_j)_j$ converges to f in $\mathcal{S}'(\mathbb{R}^n)$. Let now $0 < p < 1$. Then applying Nikol'skij type inequality (cf. [Tri83, p. 18, Remark 1 and p. 23, Remark 4], for example)

$$\begin{aligned}
\|g_{j+t} - g_j|L_1(\mathbb{R}^n)\| &\leq \|g_{j+t} - g_{j+t-1}|L_1(\mathbb{R}^n)\| + ... + \|g_{j+1} - g_j|L_1(\mathbb{R}^n)\| \\
&\leq c2^{(j+t)\varepsilon n(\frac{1}{p}-1)}\|g_{j+t} - g_{j+t-1}|L_p(\mathbb{R}^n)\| + .. + c2^{(j+1)\varepsilon n(\frac{1}{p}-1)}\|g_{j+1} - g_j|L_p(\mathbb{R}^n)\| \\
&\leq c'2^{(j+t)\varepsilon n(\frac{1}{p}-1)}\|g_{j+t} - f|L_p(\mathbb{R}^n)\| + ... + c'2^{j\varepsilon n(\frac{1}{p}-1)}\|g_j - f|L_p(\mathbb{R}^n)\|
\end{aligned} \tag{2.4.17}$$

Let $\eta \in (0, \underline{s}(\sigma)/2)$. We obtain, from (2.4.17) (applying the Hölder inequality for the case $q > 1$)

$$\|g_{j+t} - g_j|L_1(\mathbb{R}^n)\|^q \leq c\sum_{l=j}^{j+t} 2^{\varepsilon\eta lq}2^{l\varepsilon n(\frac{1}{p}-1)q}\|g_l - f|L_p(\mathbb{R}^n)\|^q. \tag{2.4.18}$$

Now, from (1.2.4) and (2.4.18),

$$\|g_{j+t} - g_j|L_1(\mathbb{R}^n)\|^q \leq c\sum_{l=j}^{j+t} \sigma^q_{\varepsilon,l+k}\|g_l - f|L_p(\mathbb{R}^n)\|^q. \tag{2.4.19}$$

By (2.4.15) and (2.4.19) we conclude that $(g_j)_j$ is a Cauchy sequence in $L_1(\mathbb{R}^n)$ and so converges in $L_1(\mathbb{R}^n)$ to, say, g. Let us prove that $f = g$. We fix an arbitrary $r > 0$ and we denote by $B_r(0)$ the ball with center 0 and radius r. Then

$$\begin{aligned}\|f-g|L_p(B_r(0))\|^p &\leq \|f-g_j|L_p(B_r(0))\|^p+\|g_j-g|L_p(B_r(0))\|^p\\ &\lesssim \|f-g_j|L_p(\mathbb{R}^n)\|^p+r^{n(1-p)}\|g_j-g|L_1(\mathbb{R}^n)\|^p. \qquad (2.4.20)\end{aligned}$$

Both terms in (2.4.20) converge to 0 as $j\to 0$ and so $f=g$ almost everywhere in $B_r(0)$. As $r>0$ is arbitrary we conclude that $f=g$ almost everywhere in $\mathbb{R}^n$. So

$$\lim_{j\to\infty} g_j = f \quad \text{in} \quad L_1(\mathbb{R}^n) \qquad (2.4.21)$$

and, consequently, also in $\mathcal{S}'(\mathbb{R}^n)$. So, for all $0<p\leq\infty$, $(g_j)_j\in\mathcal{U}_p^{N_\varepsilon}$ and $\lim_{j\to\infty} g_j=f$ in $\mathcal{S}'(\mathbb{R}^n)$. Therefore, by Theorem 2.3.2, $f\in B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$. By (2.3.1)-(2.3.2) and (2.4.15),

$$\begin{aligned}\|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|_X &\leq \sigma_{\varepsilon,k}\|g_0|L_p(\mathbb{R}^n)\|+\|(\sigma_{\varepsilon,j+k}(f-g_j))_{j\in\mathbb{N}_0}|\ell_q(L_p)\|\\ &\lesssim \|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|_E+\delta.\end{aligned}$$

Letting $\delta\to 0$ we conclude (2.4.14).

Step 2. It remains to prove that (2.4.5) can be replaced by (2.4.6). If $f\in B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$, then one has by embedding and by Step 1 that $f\in L_{\overline{p}}(\mathbb{R}^n)$ and

$$\|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|_M \leq \|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|_M^* \lesssim \|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|.$$

We now prove the reverse. We adapt Triebel's proof for the classical Besov spaces in [Tri83, Section 2.5.12]. We present the proof for $q<\infty$. The case $q=\infty$ can be proved with the usual adaptations.

In this part of the proof it is convenient to distinguish properly which number b and function Λ are being considered. So we denote by Λ the function presented in Example 1.2.14(i) with $N_j=2^j$, $j\in\mathbb{N}_0$. Let $f\in L_{\overline{p}}(\mathbb{R}^n)$ such that $\|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|_{1,\Lambda,M}<\infty$. We will prove that there is $c>0$ such that

$$\int_{|u|\leq 1/2}\left(\Lambda(2^{\varepsilon k}|u|^{-1})\omega_{2M}(f,|u|)_p\right)^q\frac{du}{|u|^n}\leq c\|f|B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|_{1,\Lambda,M}^q. \qquad (2.4.22)$$

Let $\rho = \rho_0 + \rho_1$. Then

$$e^{i\rho\cdot\xi} - 1 = e^{i\rho_0\cdot\xi}(e^{i\rho_1\cdot\xi} - 1) + e^{i\rho_0\cdot\xi} - 1. \tag{2.4.23}$$

Next we will raise (2.4.23) to the power $2M$, $M \in \mathbb{N}$, apply it to $\mathcal{F}f$ and take the inverse Fourier transform of the result in order to obtain

$$\|\Delta_\rho^{2M} f|L_p(\mathbb{R}^n)\|^q \le c\|\Delta_{\rho_0}^M f|L_p(\mathbb{R}^n)\|^q + c\|\Delta_{\rho_1}^M f|L_p(\mathbb{R}^n)\|^q. \tag{2.4.24}$$

We present the calculations to obtain (2.4.24) from (2.4.23):
For $\phi \in \mathcal{S}(\mathbb{R}^n)$, $u \in \mathbb{R}^n$

$$\begin{aligned}
\Big(\mathcal{F}\big(e^{-iu\cdot\xi}(\mathcal{F}^{-1}\phi)(\xi)\big)\Big)(z) &= c_n \int_{\mathbb{R}^n} e^{-i\xi\cdot z} e^{-iu\cdot\xi}(\mathcal{F}^{-1}\phi)(\xi)d\xi \\
&= c_n \int_{\mathbb{R}^n} e^{-i\xi\cdot(z+u)}(\mathcal{F}^{-1}\phi)(\xi)d\xi \\
&= \big(\mathcal{F}(\mathcal{F}^{-1}\phi)\big)(z+u) = \phi(z+u)
\end{aligned} \tag{2.4.25}$$

Hence,

$$\begin{aligned}
\Big\langle \Big(\mathcal{F}^{-1}\big(e^{-iu\cdot\xi}(\mathcal{F}f)(\xi)\big)\Big)(y), \phi(y)\Big\rangle &= \Big\langle f(z), \Big(\mathcal{F}\big(e^{-iu\cdot\xi}(\mathcal{F}^{-1}\phi)(\xi)\big)\Big)(z)\Big\rangle \\
&= \big\langle f(z), \phi(z+u)\big\rangle \\
&= \int_{\mathbb{R}^n} f(y-u)\phi(y)dy.
\end{aligned} \tag{2.4.26}$$

So, $\mathcal{F}^{-1}\big(e^{-iu\cdot\xi}(\mathcal{F}f)(\xi)\big)$ is a regular distribution given by $f(y-u)$, $y \in \mathbb{R}^n$.
Now, on the one hand,

$$\begin{aligned}
\big[\mathcal{F}^{-1}\big((e^{i\rho\cdot\xi} - 1)^{2M}(\mathcal{F}f)(\xi)\big)\big](y) &= \sum_{t=0}^{2M} \tbinom{2M}{t}(-1)^{2M-t}\big[\mathcal{F}^{-1}(e^{it\rho\cdot\xi}(\mathcal{F}f)(\xi))\big](y) \\
&= \sum_{t=0}^{2M} \tbinom{2M}{t}(-1)^{2M-t} f(y+t\rho) \\
&= (\Delta_\rho^{2M} f)(y).
\end{aligned} \tag{2.4.27}$$

On the other hand, by (2.4.23), and writing temporarily $\|\cdot\|_p$ instead of the usual $\|\cdot|L_p(\mathbb{R}^n)\|$,

$$\begin{aligned}
\|\Delta_\rho^{2M} f\|_p &= \|\mathcal{F}^{-1}\big((e^{i\rho\cdot\xi}-1)^{2M}(\mathcal{F}f)(\xi)\big)\|_p \\
&= \Big\|\mathcal{F}^{-1}\Big(\sum_{t=0}^{2M}\tbinom{2M}{t}e^{it\rho_0\cdot\xi}(e^{i\rho_1\cdot\xi}-1)^t(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi)\Big)\Big\|_p \\
&\le c\sum_{t=0}^{M}\big\|\mathcal{F}^{-1}\big\{e^{it\rho_0\cdot y}\big[\mathcal{F}(\mathcal{F}^{-1}(e^{i\rho_1\cdot\xi}-1)^t(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi))\big](y)\big\}\big\|_p \\
&\quad + c\sum_{t=M+1}^{2M}\big\|\mathcal{F}^{-1}\big\{e^{it\rho_0\cdot y}\big[\mathcal{F}(\mathcal{F}^{-1}(e^{i\rho_1\cdot\xi}-1)^t(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi))\big](y)\big\}\big\|_p \\
&= c\,A + c\,B
\end{aligned} \tag{2.4.28}$$

Let us estimate A and B.

$$\begin{aligned}
A &= \sum_{t=0}^{M}\big\|\Delta_{\rho_1}^t\big[\mathcal{F}^{-1}(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi)\big]\big\|_p && (2.4.29)\\
&\le c'\sum_{t=0}^{M}\big\|\mathcal{F}^{-1}(e^{i\rho_0\cdot\xi}-1)^{2M-t}(\mathcal{F}f)(\xi)\big\|_p && (2.4.30)\\
&= c'\sum_{t=0}^{M}\|\Delta_{\rho_0}^{2M-t}f\|_p && (2.4.31)\\
&\le c''\|\Delta_{\rho_0}^{M}f\|_p, && (2.4.32)
\end{aligned}$$

where we applied (2.4.27) in (2.4.29) and (2.4.31). In (2.4.30) and (2.4.32) we used the fact that given $T\in\mathbb{N}_0$ there is $c_{T,p}>0$ such that

$$\|\Delta_u^T f|L_p(\mathbb{R}^n)\|\le c_{T,p}\|f|L_p(\mathbb{R}^n)\|,\quad f\in\ L_p(\mathbb{R}^n).$$

Analogously one estimates B:

$$
\begin{aligned}
B &= c \sum_{t=M+1}^{2M} \left\| \Delta_{\rho_0}^{2M-t} \left[\mathcal{F}^{-1} (e^{i\rho_1 \cdot \xi} - 1)^t (\mathcal{F} f)(\xi) \right] \right\|_p \\
&\leq c' \sum_{t=M+1}^{2M} \left\| \mathcal{F}^{-1} (e^{i\rho_1 \cdot \xi} - 1)^t (\mathcal{F} f)(\xi) \right\|_p \\
&= c' \sum_{t=M+1}^{2M} \| \Delta_{\rho_1}^t f \|_p \\
&\leq c'' \| \Delta_{\rho_1}^M f \|_p \qquad (2.4.33)
\end{aligned}
$$

We remark that the constants involved in estimations (2.4.28)-(2.4.33) are independent of ρ, ρ_0, ρ_1 and f. So we proved (2.4.24).

Integrating (2.4.24) with respect to ρ_0 in a ball $B(0,r)$, $r > 0$, we obtain

$$
\begin{aligned}
\| \Delta_\rho^{2M} f | L_p(\mathbb{R}^n) \|^q &\lesssim \frac{1}{r^n} \int_{B(0,r)} \| \Delta_{\rho_0}^M f | L_p(\mathbb{R}^n) \|^q d\rho_0 + \frac{1}{r^n} \int_{B(0,r)} \| \Delta_{\rho-\rho_0}^M f | L_p(\mathbb{R}^n) \|^q d\rho_0 \\
&= \frac{1}{r^n} \int_{B(0,t)} \| \Delta_{\rho_0}^M f | L_p(\mathbb{R}^n) \|^q d\rho_0 + \frac{1}{r^n} \int_{B(\rho,r)} \| \Delta_\lambda^M f | L_p(\mathbb{R}^n) \|^q d\lambda.
\end{aligned}
$$

Now taking the supremum we get to

$$
\begin{aligned}
\sup_{|\rho| \leq r} \| \Delta_\rho^{2M} f | L_p(\mathbb{R}^n) \|^q &\lesssim \frac{1}{r^n} \int_{B(0,r)} \| \Delta_{\rho_0}^M f | L_p(\mathbb{R}^n) \|^q d\rho_0 + \frac{1}{r^n} \int_{B(0,2r)} \| \Delta_\lambda^M f | L_p(\mathbb{R}^n) \|^q d\lambda \\
&\lesssim \frac{1}{r^n} \int_{B(0,2r)} \| \Delta_\lambda^M f | L_p(\mathbb{R}^n) \|^q d\lambda \\
&\sim \int_{B(0,2)} \| \Delta_{\lambda r}^M f | L_p(\mathbb{R}^n) \|^q d\lambda, \qquad (2.4.34)
\end{aligned}
$$

where (2.4.34) was obtained using polar coordinates. So applying (2.4.34) we obtain

$$
\begin{aligned}
&\int_{|u| \leq 1/2} \left(\Lambda(2^{\varepsilon k} |u|^{-1}) \omega_{2M}(f, |u|)_p \right)^q \frac{du}{|u|^n} \\
&\qquad \lesssim \int_{r \leq 1/2} \left(\Lambda(2^{\varepsilon k} r^{-1}) \omega_{2M}(f, r)_p \right)^q \frac{dr}{r} \\
&\qquad \lesssim \int_{r \leq 1/2} \Lambda(2^{\varepsilon k} r^{-1})^q \int_{B(0,2)} \| \Delta_{\lambda r}^M f | L_p(\mathbb{R}^n) \|^q d\lambda \frac{dr}{r} \qquad (2.4.35)
\end{aligned}
$$

Using again polar coordinates, i.e., considering

$$\lambda = x\theta, \quad x \in (0,2), \quad \theta \in S_{n-1},$$

where S_{n-1} is the unit sphere, we obtain, from (2.4.35),

$$\begin{aligned}
&\int_{|u|\le 1/2} \big(\Lambda(2^{\varepsilon k}|u|^{-1})\omega_{2M}(f,|u|)_p\big)^q \frac{du}{|u|^n} \\
&\quad \lesssim \int_0^{1/2} \Lambda(2^{\varepsilon k}r^{-1})^q \int_0^2 \int_{S_{n-1}} \|\Delta^M_{xr\theta} f|L_p(\mathbb{R}^n)\|^q x^{n-1} d\theta dx \frac{dr}{r} \\
&\quad = \int_0^2 \int_0^{1/2} \Lambda(2^{\varepsilon k}r^{-1})^q \int_{S_{n-1}} \|\Delta^M_{xr\theta} f|L_p(\mathbb{R}^n)\|^q x^{n-1} d\theta \frac{dr}{r} dx && (2.4.36)\\
&\quad = \int_0^2 \int_0^{x/2} \Lambda(2^{\varepsilon k}y^{-1}x)^q \int_{S_{n-1}} \|\Delta^M_{y\theta} f|L_p(\mathbb{R}^n)\|^q x^{n-1} d\theta \frac{dy}{y} dx && (2.4.37)\\
&\quad \le \int_0^2 \int_0^1 \left(\frac{\Lambda(2^{\varepsilon k}y^{-1}x)}{\Lambda(2^{\varepsilon k}y^{-1})}\right)^q \Lambda(2^{\varepsilon k}y^{-1})^q \int_{S_{n-1}} \|\Delta^M_{y\theta} f|L_p(\mathbb{R}^n)\|^q x^{n-1} d\theta \frac{dy}{y} dx
\end{aligned}$$

where we applied Fubini in (2.4.36) and changed variables in (2.4.37), considering $y = rx$. Let us estimate

$$\frac{\Lambda(2^{\varepsilon k}y^{-1}x)}{\Lambda(2^{\varepsilon k}y^{-1})}, \quad 0 < x \le 2, \quad 0 < y \le 1.$$

We fix $\delta \in (0, \underline{s}(\sigma))$. As $2^{\varepsilon k}y^{-1} \ge 1$, there is $t \in \mathbb{N}_0$ such that $2^{\varepsilon k}y^{-1} \in [2^t, 2^{t+1}]$. Let us consider first the case $1 \le x \le 2$. Then

$$\frac{\Lambda(2^{\varepsilon k}y^{-1}x)}{\Lambda(2^{\varepsilon k}y^{-1})} \sim \frac{\sigma_{t+1}}{\sigma_t} \lesssim 2^{\overline{s}(\sigma)+\delta}. \tag{2.4.38}$$

Now consider $0 < x \le 1$. Then there is $j \in \mathbb{N}_0$ such that $x \in [2^{-(j+1)}, 2^{-j}]$. If $2^{\varepsilon k}y^{-1}x \ge 1$ then $t \ge j-1$ and

$$\frac{\Lambda(2^{\varepsilon k}y^{-1}x)}{\Lambda(2^{\varepsilon k}y^{-1})} \sim \frac{\sigma_{t-j+1}}{\sigma_t} \lesssim 2^{-(\underline{s}(\sigma)-\delta)j} \lesssim 1. \tag{2.4.39}$$

If $2^{\varepsilon k}y^{-1}x < 1$, then, applying the fact that Λ is the function given in Example 1.2.14(i) with $N_j = 2^j$,

$$\frac{\Lambda(2^{\varepsilon k}y^{-1}x)}{\Lambda(2^{\varepsilon k}y^{-1})} \sim \frac{\sigma_0}{\Lambda(2^{\varepsilon k}y^{-1})} \sim \frac{\sigma_0}{\sigma_t} \lesssim 2^{-(\underline{s}(\sigma)-\delta)t} \lesssim 1. \tag{2.4.40}$$

Hence, applying (2.4.38)-(2.4.40) we conclude that

$$\begin{aligned}
\int_{|u|\le 1/2} & \big(\Lambda(2^{\varepsilon k}|u|^{-1})\omega_{2M}(f,|u|)_p\big)^q \frac{du}{|u|^n} \\
&\lesssim \int_0^2 x^{n-1}dx \int_0^1 \Lambda(2^{\varepsilon k}y^{-1})^q \int_{S_{n-1}} \|\Delta^M_{y\theta} f|L_p(\mathbb{R}^n)\|^q d\theta \frac{dy}{y} \\
&\lesssim \|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|^q_{1,\Lambda,M},
\end{aligned}$$

proving (2.4.22). Now by what was proved in Step 1, $f \in B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ and

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \lesssim \|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|_{1,\Lambda,M},$$

concluding the proof.

□

We collect a few further notation which will be applied in the next proposition.

Definition 2.4.4. *Let Ω be a domain in $\mathbb{R}^n$. Let σ be an admissible sequence, $0 < p \le \infty$, $0 < q \le \infty$.*

(i) Then $B^{\sigma,N}_{p,q}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a $g \in B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ with $g|_\Omega = f$. Furthermore,

$$\|f|B^{\sigma,N}_{p,q}(\Omega)\| := \inf \|g|B^{\sigma,N}_{p,q}(\mathbb{R}^n)\|, \tag{2.4.41}$$

where the infimum is taken over all $g \in B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ such that its restriction $g|_\Omega$ to Ω coincides in $D'(\Omega)$ with f.

(ii) Then $\widetilde{B}^{\sigma,N}_{p,q}(\overline{\Omega})$ is the closed subspace of $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ given by

$$\widetilde{B}^{\sigma,N}_{p,q}(\overline{\Omega}) := \{f \in B^{\sigma,N}_{p,q}(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega}\}. \tag{2.4.42}$$

Remark 2.4.5. *These spaces are quasi-Banach spaces. If $p = q$ or $N_j = 2^j, j \in \mathbb{N}_0$ we simplify the notation according to the description given in Remark 2.1.3 for Besov spaces on $\mathbb{R}^n$. Function spaces on domains have also been studied in detail from the very beginning of the theory of function spaces in 1950s and 1960s. We refer to the books [Tri83, Chapter 3], [Tri92b, Chapter 5], [Tri01, Section 5] and [Tri06, Chapter 4].*

In the next proposition we present equivalent quasi-norms for the elements of certain subspaces of Besov spaces of generalised smoothness on $\mathbb{R}^n$. It will be an important tool to prove the adapted homogeneity property for spaces of generalised smoothness.

Proposition 2.4.6. *Let Ω be a bounded domain in $\mathbb{R}^n$. Consider an admissible sequence σ and $\Lambda \in \mathcal{A}_\sigma$. Let*

$$0 < p, q \leq \infty, \quad \underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+ \quad \textit{and} \quad \overline{s}(\sigma) < M \in \mathbb{N}.$$

Let $0 < \varepsilon \leq 1$, $N_\varepsilon = (2^{\varepsilon j})_{j \in \mathbb{N}_0}$, $b > 0$ and $k \in \mathbb{N}_0$. Then

$$\|f|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \sim \bigg(\int_{|u|\leq b} \big(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta^M_u f|L_p(\mathbb{R}^n)\|\big)^q \frac{du}{|u|^n}\bigg)^{\frac{1}{q}}, \tag{2.4.43}$$

(with the usual modification if $q = \infty$), for all $f \in \widetilde{B}^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\overline{\Omega})$. The equivalence constants in (2.4.43) *are independent of f and k.*

Proof. We present the proof for $0 < q < \infty$. The case $q = \infty$ is proved making the usual adaptations.
One of the inequalities follows immediately from Theorem 2.4.1. So we just have to prove that under the above conditions there is a positive number c such that

$$\|f|L_p(\mathbb{R}^n)\| \leq c\,\sigma^{-1}_{\varepsilon,k}\bigg(\int_{|u|\leq b} \big(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta^M_u f|L_p(\mathbb{R}^n)\|\big)^q \frac{du}{|u|^n}\bigg)^{\frac{1}{q}},$$

for all $k \in \mathbb{N}_0$ and for all $f \in \widetilde{B}^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\overline{\Omega})$. As $\operatorname{supp} f$ is contained in a fixed bounded set, for $0 < p < 1$, we have

$$\|f|L_p(\mathbb{R}^n)\| \leq c\|f|L_1(\mathbb{R}^n)\|.$$

Hence it is enough to prove that, for $0 < p \leq \infty$,

$$\|f|L_{\overline{p}}(\mathbb{R}^n)\| \leq c\,\sigma^{-1}_{\varepsilon,k}\bigg(\int_{|u|\leq b} \big(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta^M_u f|L_p(\mathbb{R}^n)\|\big)^q \frac{du}{|u|^n}\bigg)^{\frac{1}{q}},$$

for all $k \in \mathbb{N}_0$ and for all $f \in \widetilde{B}^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\overline{\Omega})$. We recall that $\overline{p} = \max\{1, p\}$.
We can suppose that $b \leq 1$. Therefore, considering $\delta \in (0, \underline{s}(\sigma) - n(\frac{1}{p} - 1)_+)$ and choosing

$J \in \mathbb{N}_0$ such that $2^{-\varepsilon J} \leq b$, we obtain

$$\begin{aligned}
&\sigma_{\varepsilon,k}^{-q} \int_{|u|\leq b} \left(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta_u^M f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n} \\
&\quad \geq \sum_{j=J}^{\infty} \int_{2^{-\varepsilon(j+1)}\leq|u|\leq 2^{-\varepsilon j}} \left(\frac{\Lambda(2^{\varepsilon k}|u|^{-1})}{\sigma_{\varepsilon,k}}\|\Delta_u^M f|L_p(\mathbb{R}^n)\|\right)^q \frac{du}{|u|^n} \\
&\quad \geq c \int_{|u|\leq 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q}\|\Delta_u^M f|L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n}, \qquad (2.4.44)
\end{aligned}$$

where we used that fact that, for $2^{-\varepsilon(j+1)} \leq |u| \leq 2^{-\varepsilon j}$,

$$\frac{\Lambda(2^{\varepsilon k}|u|^{-1})}{\sigma_{\varepsilon,k}} \sim \frac{\sigma_{\varepsilon,j+k}}{\sigma_{\varepsilon,k}} \geq c_1 2^{(\underline{s}(\sigma_\varepsilon)-\varepsilon\delta)j} \sim |u|^{-(\underline{s}(\sigma)-\delta)}.$$

By Theorem 2.4.1, applying (1.2.4) and (2.4.44) we can easily prove that

$$B_{p,q}^{T_k(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n) \subset B_{p,q}^{(\underline{s}(\sigma)-\delta)}(\mathbb{R}^n), \quad \text{for all} \quad k \in \mathbb{N}_0, \qquad (2.4.45)$$

where the constants implicitly involved are independent of k.

By (2.4.44) and (2.4.45) we just have to prove that there is a number c such that, for all $f \in \widetilde{B}_{p,q}^{(\underline{s}(\sigma)-\delta)}(\overline{\Omega})$,

$$\|f|L_{\overline{p}}(\mathbb{R}^n)\| \leq c\left(\int_{|u|\leq 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q}\|\Delta_u^M f|L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n}\right)^{\frac{1}{q}}. \qquad (2.4.46)$$

Assuming that there is no such c, then for every $j \in \mathbb{N}$ one finds a function $f_j \in \widetilde{B}_{p,q}^{(\underline{s}(\sigma)-\delta)}(\overline{\Omega})$ which can be normalised such that

$$1 = \|f_j|L_{\overline{p}}(\mathbb{R}^n)\| > j\left(\int_{|u|\leq 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q}\|\Delta_u^M f_j \,|\, L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n}\right)^{1/q}. \qquad (2.4.47)$$

We consider

$$B_R = \{x \in \mathbb{R}^n : |x| < R\}$$

with $\overline{\Omega} \subset B_R$. Hence

$$\begin{aligned}
\|f_j|_{B_R} \,|\, B_{p,q}^{(\underline{s}(\sigma)-\delta)}(B_R)\| &\leq \|f_j|B_{p,q}^{(\underline{s}(\sigma)-\delta)}(\mathbb{R}^n)\| \\
&\sim \|f_j|L_{\overline{p}}(\mathbb{R}^n)\| + \left(\int_{|u|\leq 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q}\|\Delta_u^M f_j \,|\, L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n}\right)^{1/q} \\
&< 1 + \frac{1}{j} \leq 2. \qquad (2.4.48)
\end{aligned}$$

According to (2.4.48), $\{f_j|_{B_R}\}_{j\in\mathbb{N}}$ is bounded in $B^{(\underline{s}(\sigma)-\delta)}_{p,q}(B_R)$ and so, as the embedding of $B^{(\underline{s}(\sigma)-\delta)}_{p,q}(B_R)$ into $L_{\overline{p}}(B_R)$ is compact (cf. e.g. [Tri06, Theorem 1.97, Proposition 4.6] and the references given there), $\{f_j|_{B_R}\}_{j\in\mathbb{N}}$ is precompact in $L_{\overline{p}}(B_R)$. We may assume that, for some $f \in L_{\overline{p}}(B_R)$,

$$f_j|_{B_R} \to f \quad \text{in} \quad L_{\overline{p}}(B_R), \tag{2.4.49}$$

and, consequently,

$$\|f|L_{\overline{p}}(B_R)\| = 1. \tag{2.4.50}$$

For $j, j' \in \mathbb{N}$, using (2.4.6), we get

$$\begin{aligned}
&\|f_j - f_{j'} \,|\, B^{(\underline{s}(\sigma)-\delta)}_{p,q}(\mathbb{R}^n)\| \\
&\sim \|f_j - f_{j'}|L_{\overline{p}}(\mathbb{R}^n)\| + \left(\int_{|u|\le 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q} \|\Delta^M_u (f_j - f_{j'}) \,|\, L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n}\right)^{1/q}.
\end{aligned}$$

Using (2.4.49) and the fact that $f_j \in \widetilde{B}^{(\underline{s}(\sigma)-\delta)}_{p,q}(\overline{\Omega})$, we obtain

$$\|f_j - f_{j'}|L_{\overline{p}}(\mathbb{R}^n)\| = \|f_j|_{B_R} - f_{j'}|_{B_R} \,|\, L_{\overline{p}}(B_R)\| \to 0 \quad \text{as} \quad j, j' \to \infty.$$

By (2.4.47) we get

$$\begin{aligned}
&\left(\int_{|u|\le 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q} \|\Delta^M_u (f_j - f_{j'}) \,|\, L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n}\right)^{1/q} \\
&\lesssim \left(\int_{|u|\le 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q} \|\Delta^M_u f_j \,|\, L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n}\right)^{1/q} \\
&\quad + \left(\int_{|u|\le 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q} \|\Delta^M_u f_{j'} \,|\, L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n}\right)^{1/q} \\
&< \frac{1}{j}\, \|f_j|L_{\overline{p}}(\mathbb{R}^n)\| + \frac{1}{j'}\, \|f_{j'}|L_{\overline{p}}(\mathbb{R}^n)\| \\
&= \frac{1}{j} + \frac{1}{j'} \to 0 \quad \text{as} \quad j, j' \to \infty.
\end{aligned}$$

Hence $\{f_j\}_{j\in\mathbb{N}}$ converges in $B^{(\underline{s}(\sigma)-\delta)}_{p,q}(\mathbb{R}^n)$ to, say, g. As all f_j are elements of $\widetilde{B}^{(\underline{s}(\sigma)-\delta)}_{p,q}(\overline{\Omega})$, which is a closed subspace of $B^{(\underline{s}(\sigma)-\delta)}_{p,q}(\mathbb{R}^n)$, we get $g \in \widetilde{B}^{(\underline{s}(\sigma)-\delta)}_{p,q}(\overline{\Omega})$. As

$$\|f_j - g|B^{(\underline{s}(\sigma)-\delta)}_{p,q}(\mathbb{R}^n)\| \to 0 \quad \text{as} \quad j \to \infty,$$

we get

$$\|f_j|_{B_R} - g|_{B_R}|L_{\overline{p}}(B_R)\| = \|f_j - g|L_{\overline{p}}(\mathbb{R}^n)\| \to 0 \quad \text{as} \quad j \to \infty$$

and $g = f$ according to (2.4.49). Thus, by (2.4.50),

$$\|g|_{B_R} \mid L_{\overline{p}}(B_R)\| = \|f_j|_{B_R}|L_{\overline{p}}(B_R)\| = 1. \tag{2.4.51}$$

Now (assuming $q < \infty$)

$$\begin{aligned}
&\int_{|u|\le 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q}\|\Delta_u^M g \mid L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n} \\
&\lesssim \int_{|u|\le 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q}\|\Delta_u^M (g-f_j) \mid L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n} + \int_{|u|\le 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q}\|\Delta_u^M f_j \mid L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n} \\
&\lesssim \|g-f_j|B_{p,q}^{(\underline{s}(\sigma)-\delta)}(\mathbb{R}^n)\|^q + \frac{1}{j^q} \to 0 \quad \text{as} \quad j\to\infty.
\end{aligned}$$

Therefore

$$\int_{|u|\le 2^{-\varepsilon J}} |u|^{-(\underline{s}(\sigma)-\delta)q}\|\Delta_u^M g \mid L_p(\mathbb{R}^n)\|^q \frac{du}{|u|^n} = 0. \tag{2.4.52}$$

If $q = \infty$ one has to modify the argument in the usual way.

From (2.4.52) follows that

$$(\Delta_u^M g)(x) = 0 \quad \text{for almost all} \quad |u| \le 2^{-\varepsilon J} \quad \text{and} \quad x \in \mathbb{R}^n. \tag{2.4.53}$$

Hence

$$\int_{|u|\le 2^{-\varepsilon J}} |u|^{-2s}\|\Delta_u^M g|L_v(\mathbb{R}^n)\|^2 \frac{du}{|u|^n} = 0, \quad 0 < v \le \infty, \quad s \in \mathbb{R}. \tag{2.4.54}$$

If $p \ge 2$ then $L_p(B_R) \subset L_2(B_R)$ and so, by the conditions on the support of g, $g \in L_2(\mathbb{R}^n)$. Hence, by (2.4.54) and Theorem 2.4.1, g (identified with its restriction to B_R) belongs to $B_{2,2}^{(m)}(B_R)$, for all $m \in \mathbb{N}$. If $0 < p < 2$ then by (2.4.54) and Theorem 2.4.1, g (identified with its restriction to B_R) belongs to $B_{p,2}^{(s)}(B_R)$, for all $s \in \mathbb{R}$, in particular, for all $s = m + n(\frac{1}{p} - \frac{1}{2})$, $m \in \mathbb{N}$. Therefore, by a well-known embedding (cf. [Tri83, p. 196, Theorem 3.3.1], for example), $g \in B_{2,2}^{(m)}(B_R)$, for all $m \in \mathbb{N}$. So, for all $0 < p \le \infty$, g (identified with its restriction to B_R) belongs to $B_{2,2}^{(m)}(B_R)$, for all $m \in \mathbb{N}$. As these spaces coincide with the

Sobolev spaces $W_2^m(B_R)$, (cf. [Tri83, p. 88, 2.5.6]), then $g \in C^\infty(\overline{B_R})$. This follows from [EE87, p. 241, Theorem 3.20] (we also refer to pages 202 and 222 for notation). According to [Tri06, p. 201, Remark 4.11], as $g \in C^\infty(\overline{B_R})$ and satisfies (2.4.53), then g must be (locally) polynomial of degree less than M. By compactness arguments it follows that g is globally in B_R a polynomial of degree less than M. As $\operatorname{supp} g \in \overline{\Omega} \subset B_R$, then $g = 0$. But this contradicts (2.4.51). □

Next, we present and prove an adapted homogeneity property for Besov spaces of generalised smoothness.

Theorem 2.4.7. *Let $0 < p, q \leq \infty$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+.$$

Fix $c_0 > 0$ and $0 < \varepsilon \leq 1$. Let $N_\varepsilon = (2^{\varepsilon j})_{j \in \mathbb{N}_0}$. Then

$$\|f|B^{\sigma_\varepsilon, N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \sim 2^{-\frac{\varepsilon k n}{p}} \|f(2^{-\varepsilon k}\cdot)|B^{T_k(\sigma_\varepsilon), N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \tag{2.4.55}$$

for all $k \in \mathbb{N}_0$ and all

$$f \in B^{\sigma_\varepsilon, N_\varepsilon}_{p,q}(\mathbb{R}^n) \quad \textit{with} \quad \operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x - x_0| \leq c_0 2^{-\varepsilon k}\},$$

for some $x_0 \in \mathbb{R}^n$. The equivalence constants in (2.4.55) are independent of x_0, k and f.

Proof. Step 1. Let $x_0 \in \mathbb{R}^n$ be such that $\operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x - x_0| \leq c_0 2^{-\varepsilon k}\}$. We may assume that $x_0 = 0$, otherwise we consider first $f(\cdot + x_0)$ and $f(2^{-\varepsilon k} \cdot + x_0)$ and then conclude for f and $f(2^{-\varepsilon k}\cdot)$. Let Λ be an admissible function associated to σ. Let $f \in B^{\sigma_\varepsilon, N_\varepsilon}_{p,q}(\mathbb{R}^n)$ be such that $\operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x| \leq c_0 2^{-\varepsilon k}\}$. We remark that both $\operatorname{supp} f$ and $\operatorname{supp} f(2^{-\varepsilon k}\cdot)$ are contained in $\{x \in \mathbb{R}^n : |x| < c_0\}$. Clearly, $f(2^{-\varepsilon k}\cdot) \in L_{\overline{p}}(\mathbb{R}^n)$

and $\sigma_{\varepsilon,k}\|f(2^{-\varepsilon k}\cdot)|L_p(\mathbb{R}^n)\|$ is finite. For a fixed $b>0$ and an integer M such that $M>\overline{s}(\sigma)$,

$$\begin{aligned}
&\left(\int_{|u|\le b}\big(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta_u^M(f(2^{-\varepsilon k}\cdot))|L_p(\mathbb{R}^n)\|\big)^q\frac{du}{|u|^n}\right)^{\frac{1}{q}}\\
&\quad= 2^{\frac{\varepsilon kn}{p}}\left(\int_{|u|\le b}\big(\Lambda(2^{\varepsilon k}|u|^{-1})\|\Delta_{2^{-\varepsilon k}u}^M f|L_p(\mathbb{R}^n)\|\big)^q\frac{du}{|u|^n}\right)^{\frac{1}{q}}\\
&\quad= 2^{\frac{\varepsilon kn}{p}}\left(\int_{|u|\le 2^{-\varepsilon k}b}\big(\Lambda(|u|^{-1})\|\Delta_u^M f|L_p(\mathbb{R}^n)\|\big)^q\frac{du}{|u|^n}\right)^{\frac{1}{q}}\\
&\quad\lesssim 2^{\frac{\varepsilon kn}{p}}\|f|B^{\sigma_\varepsilon,N_\varepsilon}_{p,q}(\mathbb{R}^n)\|, \qquad (2.4.56)
\end{aligned}$$

where we applied Theorem 2.4.1 in the last estimate. Again by Theorem 2.4.1 we conclude that $f(2^{-\varepsilon k}\cdot)\in B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$. Now

$$\|f(2^{-\varepsilon k}\cdot)|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|\lesssim 2^{\frac{\varepsilon kn}{p}}\|f|B^{\sigma_\varepsilon,N_\varepsilon}_{p,q}(\mathbb{R}^n)\|.$$

follows from Proposition 2.4.6 and (2.4.56).

Step 2. As $f(2^{-\varepsilon k}\cdot)\in B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$ then, by Theorem 2.2.12, there is $\nu\in b_{p,q}$ such that

$$f(2^{-\varepsilon k}x)=\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\nu_{j,l}a^{j,l}(x),$$

where $a^{j,l}$ are, respectively, d-$T_k(\sigma_\varepsilon)$-1_K-ε-atoms and d-$(T_k(\sigma_\varepsilon),p)_K$-$\varepsilon$-atoms related to the $2^{-\varepsilon j}$-approximate lattices, $j\in\mathbb{N}_0$, described in Example 2.2.3, for some conveniently chosen and fixed K and d. Moreover, ν can be chosen such that

$$\|\nu|b_{p,q}\|\le c\|f(2^{-\varepsilon k}\cdot)|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|, \qquad (2.4.57)$$

where c is a positive number independent of f and k. We can easily check that the functions defined by

$$b^{j+k,l}(x):=2^{\frac{\varepsilon kn}{p}}a^{j,l}(2^{\varepsilon k}x)$$

are d-$(\sigma_\varepsilon,p)_K$-ε-atoms. Then,

$$f(x)=\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\eta_{j+k,l}b^{j+k,l}(x),\quad\text{with}\quad \eta_{j+k,l}:=2^{-\frac{\varepsilon kn}{p}}\nu_{j,l},$$

and, therefore,

$$\|f|B^{\sigma_\varepsilon,N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \lesssim \|\eta|b_{p,q}\| = 2^{-\frac{\varepsilon kn}{p}}\|\nu|b_{p,q}\|$$

Hence, by (2.4.57), we obtain

$$\|f|B^{\sigma_\varepsilon,N_\varepsilon}_{p,q}(\mathbb{R}^n)\| \le 2^{-\frac{\varepsilon kn}{p}}c\|f(2^{-\varepsilon k}\cdot)|B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)\|.$$

This concludes the proof. □

Remark 2.4.8. *It follows immediately from the above proof that* $f \in B^{\sigma_\varepsilon,N_\varepsilon}_{p,q}(\mathbb{R}^n)$ *if, and only if,* $f(2^{-\varepsilon k}\cdot) \in B^{T_k(\sigma_\varepsilon),N_\varepsilon}_{p,q}(\mathbb{R}^n)$.

Remark 2.4.9. *Let us compare* (2.4.55) *with the homogeneity property obtained for the Besov spaces of classical smoothness* $B^{(s)}_{p,q}(\mathbb{R}^n)$ *according to Remark 2.1.3. So, we consider*

$$0 < p,q \le \infty, \quad \varepsilon = 1 \quad \textit{and} \quad \sigma = (s) \quad \textit{with} \quad s > n\Big(\frac{1}{p} - 1\Big)_+.$$

According to (2.4.55), *for all*

$$f \in B^{(s)}_{p,q}(\mathbb{R}^n) \quad \textit{with} \quad \operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x - x_0| \le c_0 2^{-k}\}, \tag{2.4.58}$$

for some $x_0 \in \mathbb{R}^n$, *we have*

$$\|f|B^{(s)}_{p,q}(\mathbb{R}^n)\| \sim 2^{-\frac{kn}{p}}\|f(2^{-k}\cdot)|B^{T_k((s))}_{p,q}(\mathbb{R}^n)\|. \tag{2.4.59}$$

For all $g \in B^{(s)}_{p,q}(\mathbb{R}^n)$,

$$\begin{aligned}\|g|B^{T_k((s))}_{p,q}(\mathbb{R}^n)\| &\sim \big\|\big(2^{(j+k)s}\varphi_j(D)g\big)_{j\in\mathbb{N}_0}|\ell_q(L_p)\big\| \\ &= 2^{ks}\big\|\big(2^{js}\varphi_j(D)g\big)_{j\in\mathbb{N}_0}|\ell_q(L_p)\big\| \\ &\sim 2^{ks}\|g|B^{(s)}_{p,q}(\mathbb{R}^n)\|,\end{aligned}$$

where $\varphi = (\varphi_j)_{j\in\mathbb{N}_0}$ *is some partition of unity according to Definition 2.1.1. Hence we can rewrite* (2.4.59). *For all* f *as in* (2.4.58),

$$\|f|B^{(s)}_{p,q}(\mathbb{R}^n)\| \sim 2^{k(s-\frac{n}{p})}\|f(2^{-k}\cdot)|B^{(s)}_{p,q}(\mathbb{R}^n)\|,$$

which corresponds to the homogeneity property for Besov spaces (cf. [CLT07]).

2.5 Characterisation by non-smooth atomic decompositions

In this subsection we present decompositions with non-smooth atoms for the elements of certain Besov spaces of generalised smoothness on $\mathbb{R}^n$. The definition we present next is a generalisation of the one introduced by Triebel in [Tri03]. We use the abbreviations according to Remark 2.1.3. In particular $B_p^{(a)\sigma}(\mathbb{R}^n) = B_{p,p}^{(a)\sigma}(\mathbb{R}^n)$ has the meaning as explained there.

Definition 2.5.1. *Let $0 < \varepsilon \leq 1$ and $0 < p \leq \infty$. For $j \in \mathbb{N}_0$, let $\{y^{j,l}\}_{l\in\mathbb{Z}^n}$ be a $2^{-\varepsilon j}$-approximate lattice as in Definition 2.2.1. Let $d > c_{\varepsilon,2}$, where $c_{\varepsilon,2}$ is as in* (2.2.2). *We fix $a > 0$, consider an admissible sequence σ and*

$$(a)\sigma := (2^{aj}\sigma_j)_{j\in\mathbb{N}_0}$$

Then $a^{j,l} \in B_p^{(a)\sigma}(\mathbb{R}^n)$ is called a d-$(\sigma,p)_a$-ε-atom if

$$\operatorname{supp} a^{j,l} \subset B(y^{j,l}, d2^{-\varepsilon j}), \qquad j \in \mathbb{N}_0, \quad l \in \mathbb{Z}^n,$$

and

$$\|a^{j,l}|B_p^{(a)\sigma}(\mathbb{R}^n)\| \leq 2^{\varepsilon a j} \tag{2.5.1}$$

The proofs of the next Proposition and Theorem follow the proofs for the classical case in [Tri03]. The most important step was to find the "substitute" for the homogeneity property, which was presented in the above subsection.

In the next Proposition we assert that non-smooth atoms are correctly normalised.

Proposition 2.5.2. *Let $0 < \varepsilon \leq 1$, $0 < p \leq \infty$, $d > c_{\varepsilon,2}$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+.$$

We fix $a > 0$ and consider $a^{j,l}$, with $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$, a d-$(\sigma,p)_a$-ε-atom according to Definition 2.5.1. Then

$$\|a^{j,l}|B_p^{\sigma}(\mathbb{R}^n)\| \lesssim 1 \qquad \text{and} \qquad \|a^{j,l}|L_p(\mathbb{R}^n)\| \lesssim \sigma_{\varepsilon,j}^{-1}.$$

Proof. Let $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$, β denote the sequence $(a)\sigma$ and Λ be an admissible function associated to σ. Then $\Lambda(2^{\varepsilon j}\cdot)(2^{\varepsilon j}\cdot)^a$ is an admissible function associated to $\{T_j(\beta_\varepsilon), N_\varepsilon\}$.

For all $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, $a^{j,l} \in B_p^\beta(\mathbb{R}^n)$. Then, by Proposition 2.1.7, $a^{j,l} \in B_p^{\beta_\varepsilon,N_\varepsilon}(\mathbb{R}^n)$ and, by Remark 2.4.8, $a^{j,l}(2^{-\varepsilon j}\cdot) \in B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$. Hence $a^{j,l}(2^{-\varepsilon j}\cdot+y^{j,l})$ belongs to $B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$ and $\operatorname{supp} a^{j,l}(2^{-\varepsilon j}\cdot+y^{j,l}) \subset \{x : |x| \le d\}$, for all $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$. As $a > 0$, the last assertion remains true if we replace β by σ. So, applying Proposition 2.4.6, we get for some $M > a + \overline{s}(\sigma)$,

$$\begin{aligned}
\|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\sigma_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|^p &\sim \int_{|u|\le 1} \Lambda(2^{\varepsilon j}|u|^{-1})^p \|\Delta_u^M f|L_p(\mathbb{R}^n)\|^p \frac{du}{|u|^n} \\
&\le\ 2^{-\varepsilon jap}\int_{|u|\le 1} \Lambda(2^{\varepsilon j}|u|^{-1})^p (2^{\varepsilon ja}|u|^{-a})^p \|\Delta_u^M f|L_p(\mathbb{R}^n)\|^p \frac{du}{|u|^n} \\
&\sim\ 2^{-\varepsilon jap}\|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|^p,
\end{aligned}$$

where the equivalence constants are independent of j. So, by Proposition 2.1.7, Theorem 2.4.7 and also by (2.5.1),

$$\begin{aligned}
\|a^{j,l}|B_p^\sigma(\mathbb{R}^n)\| &\sim \|a^{j,l}|B_p^{\sigma_\varepsilon,N_\varepsilon}(\mathbb{R}^n)\| \\
&\lesssim 2^{-\frac{\varepsilon jn}{p}} 2^{-\varepsilon ja}\|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \\
&\sim 2^{-\varepsilon ja}\|a^{j,l}|B_p^{\beta_\varepsilon,N_\varepsilon}(\mathbb{R}^n)\| \\
&\sim 2^{-\varepsilon ja}\|a^{j,l}|B_p^{\beta}(\mathbb{R}^n)\| \\
&\lesssim 1.
\end{aligned}$$

By Theorem 2.4.1, there is a positive number c, independent of j, such that

$$\|f|L_p(\mathbb{R}^n)\| \le c2^{-\varepsilon ja}\sigma_{\varepsilon,j}^{-1}\|f|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|, \quad \text{for all} \quad f \in B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n).$$

Hence

$$\|a^{j,l}|L_p(\mathbb{R}^n)\| \lesssim 2^{-\frac{\varepsilon jn}{p}} 2^{-\varepsilon ja}\sigma_{\varepsilon,j}^{-1}\|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|$$

and again by Theorem 2.4.7, Proposition 2.1.7 and (2.5.1) we conclude. $\square$

Let b_p be as in Definition 2.2.6.

Theorem 2.5.3. *Let $0 < \varepsilon \le 1$, $0 < p \le \infty$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > n\Big(\frac{1}{p} - 1\Big)_+ .$$

Let $d > c_{\varepsilon,2}$ and $a > 0$. Then $B^{\sigma}_{p}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ which can be represented as

$$f(x) = \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \nu_{j,l} a^{j,l}(x), \tag{2.5.2}$$

where $\nu = (\nu_{j,l})_{j,l} \in b_p$ and $a^{j,l}$ are d-$(\sigma,p)_a$-ε-atoms. Furthermore,

$$\|f|B^{\sigma}_{p}(\mathbb{R}^n)\| \sim \inf \|\nu|b_p\|$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (2.5.2).

Proof. Step 1. Let $K > a + \overline{s}(\sigma)$. We have Theorem 2.2.12, which guarantees that all the elements of $B^{\sigma}_{p,q}(\mathbb{R}^n)$ admit representations with d-$(\sigma_\varepsilon, p)_K$-$\varepsilon$-atoms according to Definition 2.2.10. We consider such an atom $a^{j,l}$. We can easily check that $2^{-\varepsilon j a} a^{j,l}$ is an d-$(((a)\sigma)_\varepsilon, p)_K$-$\varepsilon$-atom. Hence, by Theorem 2.2.12, we get to

$$\|a^{j,l}|B^{(a)\sigma}_{p}(\mathbb{R}^n)\| \lesssim 2^{\varepsilon a j}$$

and so, as the condition for $\operatorname{supp} a^{j,l}$ is also satisfied, we conclude that, up to constants, $a^{j,l}$ is a d-$(\sigma,p)_a$-ε-atom.

Step 2. It remains to prove that there is a number $c > 0$ such that for all f as in (2.5.2),

$$\|f|B^{\sigma}_{p}(\mathbb{R}^n)\| \le c\|\nu|b_p\|.$$

We will prove it for $0 < p < \infty$. The case $p = \infty$ is proved similarly with the usual adaptations. Let $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$. In this proof we will denote by β the sequence $(a)\sigma = (2^{aj}\sigma_j)_{j\in\mathbb{N}_0}$. For all $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, by Theorem 2.4.7, Proposition 2.1.7 and Definition 2.5.1,

$$\|a^{j,l}(2^{-\varepsilon j}\cdot)|B^{T_j(\beta_\varepsilon),N_\varepsilon}_{p}(\mathbb{R}^n)\| \sim 2^{\frac{\varepsilon j n}{p}} \|a^{j,l}|B^{\beta}_{p}(\mathbb{R}^n)\| \le 2^{\varepsilon j (a + \frac{n}{p})}. \tag{2.5.3}$$

By Theorem 2.2.12, for all $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, there is $\lambda^{j,l} \in b_p$ such that

$$a^{j,l}(2^{-\varepsilon j}x) = \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^{j,l}_{k,m} b^{k,m}_{j,l}(x),$$

with

$$c\|\lambda^{j,l}|b_p\| \leq \|a^{j,l}(2^{-\varepsilon j}\cdot)|B_p^{T_j(\beta_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\|, \tag{2.5.4}$$

where the constant c is independent of j and $b^{k,m}_{j,l}$ are d-$(T_j(\beta_\varepsilon),p)_K$-$\varepsilon$-atoms located at $d'Q_{\varepsilon k,m}$, for some $d' > 1$ fixed, with $K > a + \overline{s}(\sigma)$.

The functions defined by

$$d^{j+k,m}_{j,l}(x) := 2^{\varepsilon a(j+k)} 2^{\frac{\varepsilon jn}{p}} b^{k,m}_{j,l}(2^{\varepsilon j}x)$$

are d-$(\sigma_\varepsilon, p)_K$-$\varepsilon$-atoms located at $d'Q_{\varepsilon(j+k),m}$.

Then

$$\begin{aligned} a^{j,l}(x) &= 2^{-\varepsilon j(a+\frac{n}{p})} \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{-\varepsilon ak} \lambda^{j,l}_{k,m} d^{j+k,m}_{j,l}(x) \\ &= 2^{-\varepsilon j(a+\frac{n}{p})} \sum_{k=j}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{-\varepsilon a(k-j)} \lambda^{j,l}_{k-j,m} d^{k,m}_{j,l}(x). \end{aligned}$$

Hence

$$\begin{aligned} f &= \sum_{j=0}^{\infty} \sum_{l\in\mathbb{Z}^n} \sum_{k=j}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{-\varepsilon j\frac{n}{p}} 2^{-\varepsilon ak} \nu_{j,l} \lambda^{j,l}_{k-j,m} d^{k,m}_{j,l}(x) \chi_{B(y^{j,l},d2^{-\varepsilon j})}(x) \\ &= \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \sum_{j\leq k} \sum_{l\in\mathbb{Z}^n} 2^{-\varepsilon j\frac{n}{p}} 2^{-\varepsilon ak} \nu_{j,l} \lambda^{j,l}_{k-j,m} d^{k,m}_{j,l}(x) \chi_{B(y^{j,l},d2^{-\varepsilon j})}(x). \end{aligned}$$

For $k \in \mathbb{N}_0, m \in \mathbb{Z}^n$ and $j \leq k$ let (j,k,m) be the collection of all $l \in \mathbb{Z}^n$ such that $\operatorname{supp} a^{j,l} \cap \operatorname{supp} d^{k,m}_{j,l}$ is not empty. Each of such sets has at most M elements, where M is a natural number independent of j, k and m.Then, we get

$$f = \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \eta_{k,m} d^{k,m}(x),$$

where

$$\eta_{k,m} := \sum_{j\leq k} \sum_{l\in(j,k,m)} |\nu_{j,l}| \cdot |\lambda^{j,l}_{k-j,m}| 2^{-\varepsilon ak} 2^{-\frac{\varepsilon jn}{p}}$$

and

$$d^{k,m}(x) := \frac{\sum_{j\le k}\sum_{l\in(j,k,m)} 2^{-\frac{\varepsilon jn}{p}}\nu_{j,l}\lambda^{j,l}_{k-j,m}d^{k,m}_{j,l}(x)}{\sum_{j\le k}\sum_{l\in(j,k,m)} 2^{-\frac{\varepsilon jn}{p}}|\nu_{j,l}|\cdot|\lambda^{j,l}_{k-j,m}|}.$$

are d-$(\sigma_\varepsilon, p)_K$-$\varepsilon$-atoms located at $d'Q_{\varepsilon k,m}$. We can prove that for some fixed $\delta \in (0, \varepsilon a)$, for all $0 < p < \infty$,

$$|\eta_{k,m}|^p \le c2^{-\varepsilon akp}\sum_{j\le k}\sum_{l\in(j,k,m)}|\nu_{j,l}|^p\cdot|\lambda^{j,l}_{k-j,m}|^p 2^{(k-j)\delta p}2^{-\varepsilon jn}.$$

Therefore, by (2.5.3) and (2.5.4), we obtain

$$\|\eta|b_p\|^p \le c\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}|\nu_{j,l}|^p 2^{-\varepsilon j(n+ap)}\sum_{k=j}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda^{j,l}_{k-j,m}|^p \le c'\|\nu|b_p\|$$

and, by Theorem 2.2.12 and Proposition 2.1.7, the proof is complete. □

Remark 2.5.4. *Let us have a closer look to the type of convergence of* (2.5.2) *in Theorem 2.5.3. If $\nu \in b_p$, then* (2.5.2) *converges in $L_{\overline{p}}(\mathbb{R}^n)$. This follows from the estimations obtained in Proposition 2.5.2. Let $0 < p < \infty$ (the case $p = \infty$ is done with the usual modifications) and $\delta \in (0, \underline{s}(\sigma)\varepsilon)$. Making use of the controlled overlapping of the supports of the atoms, which follows from* (2.2.1)*, we obtain*

$$\Big\|\sum_{l\in\mathbb{Z}^n}\nu_{j,l}a^{j,l}|L_p(\mathbb{R}^n)\Big\|^p \le c\sum_{l\in\mathbb{Z}^n}|\nu_{j,l}|^p\|a^{j,l}|L_p(\mathbb{R}^n)\|^p \le c'2^{-(\underline{s}(\sigma)\varepsilon-\delta)jp}\sum_{l\in\mathbb{Z}^n}|\nu_{j,l}|^p.$$

Let

$$f_T := \sum_{j=0}^{T}\sum_{l\in\mathbb{Z}^n}\nu_{j,l}a^{j,l}.$$

Then for $T, M \in \mathbb{N}_0$, with $M < T$, and writing $\widetilde{p} = \min\{1, p\}$

$$\|f_T - f_M|L_p(\mathbb{R}^n)\|^{\widetilde{p}} \le \sum_{j=M+1}^{T}\Big\|\sum_{l\in\mathbb{Z}^n}\nu_{j,l}a^{j,l}|L_p(\mathbb{R}^n)\Big\|^{\widetilde{p}} \le c2^{-M(\underline{s}(\sigma)\varepsilon-\delta)\widetilde{p}}\|\nu|b_p\|^{\widetilde{p}}$$

and so $(f_T)_T$ converges in $L_p(\mathbb{R}^n)$. If $0 < p < 1$, then, for all admissible sequences σ,

$$B^\sigma_p(\mathbb{R}^n) \subset B^{\sigma(n)^{1-\frac{1}{p}}}_1(\mathbb{R}^n).$$

This follows from [Bri01, p. 56. 2.2.16, 2.2.17]. So, if $a^{j,l}$ are d-$(\sigma,p)_a$-ε-atoms, they also are d-$(\sigma(n)^{1-\frac{1}{p}}, 1)_a$-$\varepsilon$-atoms and then, from the previous calculations, it follows that (2.5.2) *converges in $L_1(\mathbb{R}^n)$.*

Chapter 3

Besov spaces of generalised smoothness on h-sets

In this chapter we consider Besov spaces of generalised smoothness on a special class of fractal sets in $\mathbb{R}^n$: the h-sets. First we collect some notions and properties concerning these sets. Then we define, via traces, function spaces on these fractals. We define and study an extension operator acting in a class of function spaces on dilations of h-sets and we study characterisations of Besov spaces on h-sets by smooth and non-smooth atomic decompositions.

3.1 h-sets: definition and properties

In this section we define h-sets and present some of their properties. In the designation *h-set*, the h denotes a function, which shall be a gauge function. We refer to Definition 1.4.1.

Notation 3.1.1. *In what follows, for $h \in \mathbb{H}$ and $\alpha > 0$, we denote by $\boldsymbol{h}_\alpha$ the sequence*

$$\boldsymbol{h}_\alpha := (h(2^{-\alpha j}))_{j \in \mathbb{N}_0}. \tag{3.1.1}$$

If $\alpha = 1$ we shall write only $\boldsymbol{h}$.

Definition 3.1.2. *Let $h \in \mathbb{H}$ and Γ be a non-empty compact set of $\mathbb{R}^n$. We say that Γ is an h-set if there exists a finite Radon measure μ such that*

$$\operatorname{supp} \mu = \Gamma$$

and

$$\mu(B(\gamma, r)) \sim h(r), \quad 0 < r \leq 1,\ \gamma \in \Gamma.$$

Then we say that h is a measure function (in $\mathbb{R}^n$) and that μ is an h-measure (related to Γ).

Remark 3.1.3. *The h-measures are also designated by isotropic measures (cf. [Tri06, p. 95]).*
If the function h is given by

$$h(r) = r^d \psi(r), \quad 0 < r \leq 1,$$

where $0 < d \leq n$ and $\psi : (0, 1] \to \mathbb{R}^+$ is a monotone function such that

$$\psi(2^{-j}) \sim \psi(2^{-2j}), \quad \textit{for all } j \in \mathbb{N}_0,$$

then we say that Γ is a (d, ψ)-set. If, additionally, $\psi \sim 1$ then we say that Γ is a d-set. Therefore the class of h-sets is a generalisation of the class of (d, ψ)-sets, which is itself a generalisation of d-sets. There are many authors studying these classes of fractal sets, both in fractal geometry and in the theory of function spaces. In the case of d-sets we refer to [JW84], [Mat95] and [Tri97] for example. For (d, ψ)-sets we refer to [ET98], [Mou01a] and [Mou01b]. As far as h-sets are concerned we refer to [Jon94], [Bri03], [Bri04] and [KZ06].
Some well-known self-similar fractals are examples of d-sets, namely the Cantor set in $\mathbb{R}^1$ is a d-set for $d = \log 2/\log 3$ and the von Koch curve in $\mathbb{R}^2$ is a d-set for $d = \log 4/\log 3$.

Bricchi characterised in [Bri03] which functions h are measure functions with the following outcome.

Theorem 3.1.4. *Let $h \in \mathbb{H}$. Then h is a measure function in $\mathbb{R}^n$ if, and only if, there exists a gauge function $\widetilde{h} \sim h$ such that*

$$\frac{\tilde{h}(2^{-(j+k)})}{\tilde{h}(2^{-j})} \geq 2^{-kn}, \quad j, k \in \mathbb{N}_0.$$

Remark 3.1.5. *In the case of d-sets the result above reads as follows: let $n \in \mathbb{N}$ and $d > 0$. Then there exists a d-set if and only if $d \leq n$.*

Shifting from d-sets and (d, ψ)-sets to h-sets it is often convenient to find appropriate numbers to play the role of the number d. In our results we usually consider the lower and upper Boyd indices of the sequence $\boldsymbol{h}$ given in (3.1.1) with $\alpha = 1$. Next we present some other indices, considered namely in [Bri01].

Definition 3.1.6. *Let $h \in \mathbb{H}$. We define the upper and the lower orders of h, respectively, by*

$$\overline{\omega}(h) = \limsup_{r \to 0} \frac{\log h(r)}{\log r} \quad \textit{and} \quad \underline{\omega}(h) = \liminf_{r \to 0} \frac{\log h(r)}{\log r}.$$

The results in the next proposition were proved in [Bri01]. In the proposition there is reference to Hausdorff measure and Hausdorff dimension. We refer to Definitions 1.4.2 and 1.4.4.

Proposition 3.1.7. *Let $h \in \mathbb{H}$ and $n \in \mathbb{N}$. Consider an h-set Γ and μ an h-measure (related to Γ). Then*

(i) The measure μ is equivalent to $\mathcal{H}^h_\Gamma$, where $\mathcal{H}^h_\Gamma$ is the restriction of the Hausdorff measure $\mathcal{H}^h$ in $\mathbb{R}^n$ to Γ.

(ii) The Hausdorff dimension of μ coincides with the lower order of h, $\underline{\omega}(h)$.

Remark 3.1.8. *The property in (i) was stated in [Bri01, p. 22, Theorem 1.7.6] and the property (ii) in [Bri01, p. 29, Theorem 1.8.2].*

We now present a definition about a geometric property of sets. It is useful when working with traces on Besov spaces on $\mathbb{R}^n$.

Definition 3.1.9. *A non-empty Borel set* Γ *satisfies the ball condition (or porosity condition) if there exists a number* $0 < \eta < 1$ *with the following property:*
for any ball $B(\gamma, r)$ *with* $\gamma \in \Gamma$ *and* $0 < r \leq 1$ *there is a ball* $B(x, \eta r)$ *centred at* $x \in \mathbb{R}^n$ *such that*

$$B(x, \eta\, r) \subset B(\gamma, r) \quad \textit{and} \quad B(x, \eta r) \cap \overline{\Gamma} = \emptyset.$$

The next theorem can be found in [Tri01, pp. 139-140, Proposition 9.18] and states a necessary and sufficient condition for an h-set to satisfy the ball condition.

Theorem 3.1.10. *Let* $\Gamma \subset \mathbb{R}^n$ *be an* h*-set. Then* Γ *satisfies the ball condition if, and only if, there are two positive constants* c *and* δ *such that*

$$h(2^{-\nu}) \leq c\, 2^{(n-\delta)\varkappa} h(2^{-\nu-\varkappa}), \quad \textit{for all } \nu, \varkappa \in \mathbb{N}_0.$$

The next corollary follows immediately from Theorem 3.1.10 and (1.2.4).

Corollary 3.1.11. *Let* $\Gamma \subset \mathbb{R}^n$ *be an* h*-set. Then* Γ *satisfies the ball condition if, and only if,*

$$\underline{s}(\boldsymbol{h}) > -n, \tag{3.1.2}$$

with $\underline{s}(\boldsymbol{h})$ *according to Definitions 1.2.4 and* (3.1.1).

Remark 3.1.12. *If* $h(r) = r^d$, $r > 0$, *then* (3.1.2) *is equivalent to* $d < n$.

3.2 Traces and Besov spaces on h-sets

In this section we define the operator trace and we apply it to define Besov spaces of generalised smoothness on h-sets.
Let μ be an h-measure in $\mathbb{R}^n$ according to Definition 3.1.2. For $0 < p \leq \infty$, $L_p(\Gamma, \mu)$ (or simply $L_p(\Gamma)$) denotes the usual complex quasi-Banach space (Banach if $p \geq 1$) with respect to the related measure μ, quasi-normed by

$$\|f|L_p(\Gamma, \mu)\| := \Big(\int_\Gamma |f(\gamma)|^p d\mu(\gamma) \Big)^{1/p},$$

with the usual modification if $p = \infty$.

Definition 3.2.1. *Let Γ be an h-set and let us fix an admissible sequence σ. Let $0 < p, q < \infty$. Suppose that there exists a positive constant c such that*

$$\|\varphi|_\Gamma \,|\, L_p(\Gamma)\| \le c\,\|\varphi \,|\, B^\sigma_{p,q}(\mathbb{R}^n)\|, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{3.2.1}$$

Let us consider $f \in B^\sigma_{p,q}(\mathbb{R}^n)$. As $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^\sigma_{p,q}(\mathbb{R}^n)$, there is a sequence $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_j \to f, \quad \text{as } j \to \infty, \quad \text{in } B^\sigma_{p,q}(\mathbb{R}^n).$$

By (3.2.1) *the sequence $\{\varphi_j|_\Gamma\}_{j\in\mathbb{N}_0}$ converges in $L_p(\Gamma)$ to an element which we call trace of f and we denote by $\mathrm{tr}_\Gamma f$.*

The results stated in the next proposition were proved in [Bri01].

Proposition 3.2.2. *Let $h \in \mathbb{H}$ and Γ be an h-set in $\mathbb{R}^n$.*

(i) Let $0 < p < \infty$ and $0 < q \le \min(1,p)$. Then

$$\mathrm{tr}_\Gamma : B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,q}(\mathbb{R}^n) \to L_p(\Gamma,\mu).$$

If, additionally, Γ satisfies the ball condition, then

$$\mathrm{tr}_\Gamma\, B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,q}(\mathbb{R}^n) = L_p(\Gamma,\mu).$$

(ii) Let $0 < p < \infty$, $0 < q \le \infty$ and σ be an admissible sequence with $\underline{s}(\sigma) > 0$. Then

$$\mathrm{tr}_\Gamma : B^{\sigma\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,q}(\mathbb{R}^n) \to L_p(\Gamma,\mu).$$

Remark 3.2.3. *Assertion (i) was stated and proved in [Bri01, p. 99-102, Theorem 3.3.1] where it was proved that* (3.2.1) *holds for the spaces in (i). Assertion (ii) follows from (i) and from the following inclusion*

$$B^{\sigma\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,q}(\mathbb{R}^n) \subset B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,\min(1,p)}(\mathbb{R}^n).$$

Proposition 3.2.2 can be extended to $p = \infty$, because if $p = \infty$ and $\underline{s}(\sigma) > 0$ then

$$B^{\sigma}_{\infty,q}(\mathbb{R}^n) \subset B^{(0)}_{\infty,1}(\mathbb{R}^n) \subset \mathcal{C}(\mathbb{R}^n),$$

where $\mathcal{C}(\mathbb{R}^n)$ is the space of all bounded and uniformly continuous functions in $\mathbb{R}^n$, normed in the usual way, the trace of $f \in B^{\sigma}_{\infty,q}(\mathbb{R}^n)$ being then defined as the pointwise restriction.

Next we define Besov spaces on h-sets.

Definition 3.2.4. *Consider an h-set $\Gamma \subset \mathbb{R}^n$ satisfying the ball condition. Let σ be an admissible sequence, with $\underline{s}(\sigma) > 0$ and let $0 < p, q \leq \infty$. Then we define*

$$\mathbb{B}^{\sigma}_{p,q}(\Gamma) = \mathrm{tr}_{\Gamma}\ B^{\sigma \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n) \tag{3.2.2}$$

endowed with the quasi-norm

$$\|f \,|\, \mathbb{B}^{\sigma}_{p,q}(\Gamma)\| = \inf \|g \,|\, B^{\sigma \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in B^{\sigma\, \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)$ such that $\mathrm{tr}_{\Gamma}\, g = f$.
If $p = q$ we denote these spaces by $\mathbb{B}^{\sigma}_{p}(\Gamma)$.

Remark 3.2.5. *We recall that*

$$\sigma\, \boldsymbol{h}^{1/p}(n)^{1/p} = (\sigma_j h(2^{-j})^{\frac{1}{p}} 2^{\frac{jn}{p}})_{j \in \mathbb{N}_0}.$$

The above definition was given in [Bri01, Chapter 3], where Bricchi showed that the definition makes sense and that, if we apply it to $\sigma = (0)$, we get that

$$\mathbb{B}^{(0)}_{p,q}(\Gamma) = L_p(\Gamma), \quad 0 < p < \infty, \quad 0 < q \leq \min\{1, p\}.$$

In Definition 3.2.1, tr_{Γ} was defined just for $0 < p, q < \infty$. But, if $\underline{s}(\sigma) > 0$,

$$B^{\sigma \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n) \subset B^{\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,\min(1,p)}(\mathbb{R}^n)$$

and the trace is well-defined in the space on the right, so the above definition also makes sense for $q = \infty$ and $0 < p < \infty$. If $p = \infty$ then, according to the explanation given in

Remark 3.2.3, the trace space is also well-defined.

Bricchi uses of the letter "$\mathbb{B}$", following the notation used by Triebel in [Tri97] for Besov spaces on d-sets. In [Tri97] in the definition of d-set it is not assumed that the set is compact. So, if $d, n \in \mathbb{N}$ and $d < n$, $\mathbb{R}^d$ is a d-set in $\mathbb{R}^n$. So there was already a definition for Besov spaces on $\mathbb{R}^d$ that, in some conditions, do not coincide with

$$\mathbb{B}^{(s)}_{p,q}(\mathbb{R}^d) = \mathrm{tr}_{\mathbb{R}^d}\ B^{\left(s+\frac{n-d}{p}\right)}_{p,q}(\mathbb{R}^n).$$

More details about this may be found in [Tri97, p. 160].

Example 3.2.6. *Let us consider the following particular case where we have Besov spaces on (d, ψ)-sets, according to Remark 3.1.3. So we have*

$$h(r) \sim r^d \psi(r), \quad 0 < r \leq 1,$$

and, for $s > 0$ and $a \in \mathbb{R}$, we consider

$$\sigma_j = 2^{js}\psi(2^{-j})^a, \quad \textit{for any } j \in \mathbb{N}_0.$$

In [Mou01a, p. 92], Moura defined

$$\mathbb{B}^{(s,\psi^a)}_{p,q}(\Gamma) = \mathrm{tr}_\Gamma\ B^{(s+\frac{n-d}{p},\psi^{1/p+a})}_{p,q}(\mathbb{R}^n).$$

This case is included in Definition 3.2.4. We have

$$\sigma \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}} = \left(2^{js}\psi(2^{-j})^a 2^{-\frac{jd}{p}}\psi(2^{-j})^{\frac{1}{p}}2^{\frac{jn}{p}}\right)_{j\in\mathbb{N}_0} = \left(s + \frac{n-d}{p},\, \psi^{\frac{1}{p}+a}\right).$$

If we consider additionally $\psi \sim 1$ then we get Besov spaces with classical smoothness on d-sets.

3.3 Characterisation by smooth atomic decompositions

Let Γ be a compact set in $\mathbb{R}^n$ and $\delta > 0$. Then

$$\Gamma_\delta = \{x \in \mathbb{R}^n : dist\,(x, \Gamma) < \delta\}$$

denotes a δ-neighbourhood of Γ.

Definition 3.3.1. *Let $0 < \varepsilon \leq 1$ and $j \in \mathbb{N}_0$. We say that*

$$\{\gamma^{j,m} : m = 1, ..., M_j\} \subset \Gamma \tag{3.3.1}$$

are $2^{-\varepsilon j}$-approximate lattices for Γ if there exist positive numbers $c_{\varepsilon,1}$, $c_{\varepsilon,2}$ and $c_{\varepsilon,3}$ with

$$|\gamma^{j,m_1} - \gamma^{j,m_2}| \geq c_{\varepsilon,1}\, 2^{-\varepsilon j}, j \in \mathbb{N}_0, m_1 \neq m_2, \tag{3.3.2}$$

and

$$\Gamma_{\delta_j} \subset \bigcup_{m=1}^{M_j} B(\gamma^{j,m}, c_{\varepsilon,2} 2^{-\varepsilon j}), \quad j \in \mathbb{N}_0, \tag{3.3.3}$$

where $\delta_j = c_{\varepsilon,3} 2^{-\varepsilon j}$.

Remark 3.3.2. *If Γ is an h-set, then, for $j \in \mathbb{N}_0$,*

$$M_j \sim h(2^{-\varepsilon j})^{-1}.$$

This can be proved applying [Bri01, p. 30, Lemma 1.8.3].

The approximate lattices for Γ, $\{\gamma^{j,m}\}_{m=1}^{M_j}$, $j \in \mathbb{N}_0$, can be extended to approximate lattices in $\mathbb{R}^n$ as in Definition 2.2.1.
So, using the notation of the above mentioned definition, we have that for any $j \in \mathbb{N}_0$ there is $L_j = \{l_{j,1}, ..., l_{j,M_j}\} \subset \mathbb{Z}^n$ such that

$$y^{j,l_{j,m}} = \gamma^{j,m}, \quad m = 1, ..., M_j.$$

Assumption 3.3.3. *Let $0 < \varepsilon \leq 1$ and Γ be an h-set in $\mathbb{R}^n$. For all $j \in \mathbb{N}_0$, we will denote by*

$$\{\gamma_{j,m}\}_{m=1}^{M_j} \quad \textit{and} \quad \{\delta_{j,t}\}_{t=1}^{T_j}$$

$2^{-\varepsilon j}$ and 2^{-j}-approximate lattices, respectively, for Γ, according to (3.3.1)-(3.3.3). In what follows, in all results involving approximate lattices, we assume that they and their extensions to corresponding approximate lattices in $\mathbb{R}^n$ have been fixed.

We have to adapt previous definitions:

Definition 3.3.4. *Let $0 < p, q \leq \infty$ and*

$$\lambda = \{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m = 1, ..., R_j\}$$

Then we define

$$b^{\Gamma}_{p,q} = \Big\{\lambda : \|\lambda \,|\, b^{\Gamma}_{p,q}\| = \Big(\sum_{j=0}^{\infty}\Big(\sum_{m=1}^{R_j}|\lambda_{j,m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} < \infty\Big\},$$

with the usual modification if $p = \infty$ or $q = \infty$ and with the abbreviation b^{Γ}_p if $p = q$.

Definition 3.3.5. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set fulfilling the ball condition. Let σ be an admissible sequence, $0 < \varepsilon \leq 1$, $0 < p \leq \infty$, $K \in \mathbb{N}_0$ and $d > c_{\varepsilon,2}$. Then a function $a \in \mathcal{C}^K(\mathbb{R}^n)$ is called a d-$(\sigma, p)^{\Gamma}_K$-ε-atom if*

(a) $\operatorname{supp} a \subset B(\gamma^{j,m}, d2^{-\varepsilon j})$, *for some $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$;*

(b) $\sup_{x\in\mathbb{R}^n} |D^{\alpha}a(x)| \leq \sigma_j^{-1}\, h(2^{-\varepsilon j})^{-\frac{1}{p}}\, 2^{|\alpha|\varepsilon j}$, *for $|\alpha| \leq K$.*

Bricchi obtained decompositions with this kind of atoms for the elements of Besov spaces on h-sets in the particular case where $\varepsilon = 1$ and taking atoms located in cubes $dQ_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $d > 1$ (cf. [Bri01, p. 117]). The following theorem is just an adaptation of that result, obtained from an analogous characterisation for Besov spaces on $\mathbb{R}^n$ presented in Theorem 2.2.12.

Theorem 3.3.6. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set. Let σ be an admissible sequence, $0 < p, q \leq \infty$ and $0 < \varepsilon \leq 1$. We consider*

$$\underline{s}(\boldsymbol{h}) > -n \quad \text{and} \quad \underline{s}(\sigma) > -\underline{s}(\boldsymbol{h})\Big(\frac{1}{p} - 1\Big)_+. \tag{3.3.4}$$

Let $d > c_{\varepsilon,2}$ and $K > \overline{s}(\sigma \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}})$. Then $\mathbb{B}^{\sigma}_{p,q}(\Gamma)$ is the collection of all $f \in L_p(\Gamma)$ such that

$$f = \sum_{j=0}^{\infty}\sum_{m=1}^{M_j} \nu_{j,m} a^{j,m}(x), \qquad \text{in } L_p(\Gamma), \tag{3.3.5}$$

for some $\nu \in b^{\Gamma}_{p,q}$ *and some family of* d-$(\sigma_\varepsilon, p)^{\Gamma}_{K}$-$\varepsilon$-*atoms* $a^{j,m}$. *Furthermore,*

$$\|f|\mathbb{B}^{\sigma}_{p,q}(\Gamma)\| \sim \inf \|\nu|b^{\Gamma}_{p,q}\|,$$

where the infimum is taken over all representations (3.3.5).

Remark 3.3.7. *Condition* (3.3.4) *is used to guarantee that*

$$\underline{s}(\sigma \boldsymbol{h}^{1/p}(n)^{1/p}) > n\Big(\frac{1}{p} - 1\Big)_+,$$

so that Theorem 2.2.12 can be applied to the spaces $B^{\sigma \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)$. *This possibility and also* (3.2.2) *were used by Bricchi to obtain Theorem 3.3.6.*
As it was mentioned in Corollary 3.1.11, the assumption $\underline{s}(\boldsymbol{h}) > -n$ *implies that* Γ *fulfills the ball condition.*

3.4 An extension operator

In this section we consider auxiliary function spaces on convenient dilations of a given h-set. We construct and study an extension operator acting from these function spaces into convenient Besov spaces on $\mathbb{R}^n$. We consider dilations of the h-set and, as in Chapter 2, we consider sequences $T_k(\sigma_\varepsilon)$ (we refer to Proposition 1.2.15 and Remark1.2.16). The idea is to get a kind of homogeneity property adapted to these function spaces on h-sets and to combine it with the homogeneity property for Besov spaces on $\mathbb{R}^n$ obtained in Chapter 2, in particular in Theorem 2.4.7.
Let $h \in \mathbb{H}$ and $\Gamma \subset \mathbb{R}^n$ be an h-set. Assume that

$$-\underline{s}(\boldsymbol{h}) < n \qquad \text{and} \qquad -\overline{s}(\boldsymbol{h}) > 0, \tag{3.4.1}$$

which corresponds to the condition $0 < d < n$ in the particular case of d-sets. According to Corollary 3.1.11, Γ will then fulfill the ball condition. Let $\delta \in (0, -\overline{s}(\boldsymbol{h}))$. By Definition 1.2.4, there exist positive numbers c_δ and c'_δ such that

$$c_\delta \lambda^{-\underline{s}(\boldsymbol{h})+\delta} \leq \frac{h(\lambda t)}{h(t)} \leq c'_\delta \lambda^{-\overline{s}(\boldsymbol{h})-\delta}, \qquad 0 < \lambda, t \leq 1. \tag{3.4.2}$$

This kind of condition was considered in [Jon94] and [KZ06]. In these papers Besov spaces on h-sets were defined following the same kind of approach of Jonsson and Wallin in [JW84] for d-sets. They proved the existence of an extension operator for convenient spaces on $\mathbb{R}^n$ and of a restriction operator from these spaces back to function spaces on h-sets. In [Jon94], Jonsson considered Besov spaces on h-sets which are the trace of spaces on $\mathbb{R}^n$ with classical smoothness, i.e., spaces

$$\mathbb{B}_p^{(s)\boldsymbol{h}^{-\frac{1}{p}}(n)^{-\frac{1}{p}}}(\Gamma) = \mathrm{tr}_\Gamma\, B_p^{(s)}(\mathbb{R}^n),$$

where s is a positive real number satisfying certain conditions in connection with (3.4.2) (cf. [Jon94, p.357, Theorem 1]). In [KZ06], Knopova and Zähle considered spaces of generalised smoothness

$$\mathbb{B}_p^{\tau\boldsymbol{h}^{-\frac{1}{p}}(n)^{-\frac{1}{p}}}(\Gamma) = \mathrm{tr}_\Gamma\, B_p^{\tau}(\mathbb{R}^n),$$

with $\tau_j = f(2^{2j})^{\frac{\alpha}{2}}$, $j \in \mathbb{N}_0$, where f are Bernstein functions satisfying a list of conditions and α satisfies conditions also related to (3.4.2) (cf. [KZ06, Theorem 18]). Most of the conditions considered for the functions f were applied to prove the existence and continuity of the restriction operator.

For our purposes it is convenient not to have so strong conditions for the class of sequences considered. Furthermore, on the one hand we only need to work with extension operators. On the other hand it is adequate for our work to consider these extension operators acting in convenient function spaces on a class of fractals obtained as dilations (depending on a natural number k) of a fixed h-set and taking sequences $T_k(\sigma)$ for the smoothness (guaranteeing the independence of k in all our results). The idea of considering such function spaces is to apply a kind of homogeneity now in the context of function spaces in fractals. So, we will take extension operators defined analogously but acting in a different scale of function spaces.

Remark 3.4.1. *In all calculations involved in this section and all estimations obtained we want to guarantee that the constants involved are independent of k. If nothing is said, that will be the case: the constants are independent of k.*

We introduce some more notation.

Assumption 3.4.2. *Let $h \in \mathbb{H}$ be such that* (3.4.1) *holds and let $\Gamma \subset \mathbb{R}^n$ be an h-set. We consider $c_1, c_2 > 0$ such that*

$$c_1 h(r) \leq \mu(B(\gamma, r)) \leq c_2 h(r), \quad \gamma \in \Gamma, \quad 0 < r \leq 1. \tag{3.4.3}$$

We assume that h satisfies (3.4.1) *and, consequently,* (3.4.2).

Let, for $r > 0$,

$$B^\Gamma(r) = \{\gamma \in \Gamma : |\gamma - \gamma^0| < r\}, \quad \textit{for some} \quad \gamma^0 \in \Gamma. \tag{3.4.4}$$

For $0 < \varepsilon \leq 1$ and $k \in \mathbb{N}_0$, let

$$D_{\varepsilon k} : x \mapsto 2^{\varepsilon k} x, \quad x \in \mathbb{R}^n.$$

We define

$$\mathbf{F}_{\varepsilon k} := D_{\varepsilon k} \Gamma \tag{3.4.5}$$

and

$$\Gamma_{\varepsilon k} := D_{\varepsilon k} B^\Gamma(2c_0 2^{-\varepsilon k}). \tag{3.4.6}$$

We consider the image measure

$$\mu^{\varepsilon k} := \mu \circ D_{\varepsilon k}^{-1}.$$

Set also

$$\mu_{\varepsilon k} := \frac{\mu^{\varepsilon k}}{h(2^{-\varepsilon k})} \quad \textit{and} \quad h^*_{\varepsilon,k}(r) := \frac{h(2^{-\varepsilon k} r)}{h(2^{-\varepsilon k})}, \quad r > 0. \tag{3.4.7}$$

If $\widetilde{\gamma} = 2^{\varepsilon k}\gamma$ then, for all $r > 0$,

$$\mu^{\varepsilon k}(B(\widetilde{\gamma}, r)) = \mu(B(\gamma, 2^{-\varepsilon k} r)).$$

Therefore

$$c_1 h^*_{\varepsilon,k}(r) \leq \mu_{\varepsilon k}(B(\widetilde{\gamma}, r)) \leq c_2 h^*_{\varepsilon,k}(r), \quad \widetilde{\gamma} \in \mathbf{F}_{\varepsilon k}, \quad 0 < r \leq 1, \tag{3.4.8}$$

where c_1 and c_2 are the same as in (3.4.3) and, so, independent of k.

There are $c_3, c_4 > 0$, also independent of k, such that

$$c_3 \le \mu_{\varepsilon k}(\Gamma_{\varepsilon k}) \le c_4. \tag{3.4.9}$$

Moreover, (3.4.2) implies that

$$c_\delta \lambda^{-\underline{s}(\boldsymbol{h})+\delta} \le \frac{h^*_{\varepsilon,k}(\lambda t)}{h^*_{\varepsilon,k}(t)} \le c'_\delta \lambda^{-\overline{s}(\boldsymbol{h})-\delta}, \qquad 0 < \lambda, t \le 1. \tag{3.4.10}$$

Next we define the function spaces where the extension operators will act.

Definition 3.4.3. *Consider the conditions described in Assumption 3.4.2. Let $0 < \varepsilon \le 1$, $N_\varepsilon = (2^{\varepsilon j})_{j \in \mathbb{N}_0}$, $1 \le p \le \infty$, σ be an admissible sequence and Λ be an admissible function associated to σ. We denote by $\widetilde{B}_p^{T_k(\sigma_\varepsilon),N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})$ the collection of all $\widetilde{u}$ such that*

$$\widetilde{u} \in L_p(\mathbf{F}_{\varepsilon k}, \mu_{\varepsilon k}), \quad \operatorname{supp} \widetilde{u} \subset D_{\varepsilon k} B^\Gamma(c_0 2^{-\varepsilon k}) \tag{3.4.11}$$

and, for $1 \le p < \infty$,

$$\begin{aligned} \|\widetilde{u}|\widetilde{B}_p^{T_k(\sigma_\varepsilon),N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| \ &:= \ \sigma_{\varepsilon,k}\|\widetilde{u}|L_p(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| \\ &+ \Big(\int_{\Gamma_{\varepsilon k}}\int_{\Gamma_{\varepsilon k}} \frac{\Lambda(2^{\varepsilon k}|t-v|^{-1})^p}{h^*_{\varepsilon,k}(|t-v|)} |\widetilde{u}(t) - \widetilde{u}(v)|^p d\mu_{\varepsilon k}(t) d\mu_{\varepsilon k}(v)\Big)^{1/p} \end{aligned} \tag{3.4.12}$$

is finite, or, for $p = \infty$,

$$\|\widetilde{u}|\widetilde{B}_\infty^{T_k(\sigma_\varepsilon),N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| := \sigma_{\varepsilon,k}\|\widetilde{u}|L_\infty(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| + \sup_{t,v \in \Gamma_{\varepsilon k}} \Lambda(2^{\varepsilon k}|t-v|^{-1})|\widetilde{u}(t) - \widetilde{u}(v)| \tag{3.4.13}$$

is finite.

As in [JW84], [Jon94] and [KZ06] the extension operator is constructed based in a *Whitney decomposition* of convenient sets and a kind of subordinated partition of unity. We present now the definitions and properties associated to these decompositions and functions, which are based in [Ste70].

In what follows by *cube* we mean a closed cube in $\mathbb{R}^n$, with sides parallel to the axes.

Definition 3.4.4. *We will consider $\mathcal{Q} = \{Q_i\}_i$ a numerable collection of cubes such that*

(i) $\bigcup_i Q_i = \mathbb{R}^n \setminus \mathbf{F}_{\varepsilon k}$*;*

(ii) The interiors of the cubes Q_i are mutually disjoint;

(iii) $\operatorname{diam}(Q_i) \le \operatorname{dist}(Q_i, \mathbf{F}_{\varepsilon k}) \le 4 \operatorname{diam}(Q_i)$*;*

(iv) If Q_i and Q_j touch then

$$\frac{1}{4} \operatorname{diam}(Q_i) \le \operatorname{diam}(Q_j) \le 4 \operatorname{diam}(Q_j);$$

(v) For each $Q_i \in \mathcal{Q}$ there is $t \in \mathbb{Z}$ such that the side length of Q_i is 2^{-t};

(vi) There is $N \in \mathbb{N}$ (depending only on n) such that, given $Q_i \in \mathcal{Q}$, there are at most N cubes in $\mathcal{Q}$ which touch Q_i.

Remark 3.4.5. *We refer to [Ste70, p. 167-170] where we can find a proof of the existence of such a decomposition. Hereafter we will assume that for each $k \in \mathbb{N}_0$ the decomposition described in Definition 3.4.4 is fixed and, as we shall take advantage of some properties proved by Stein, we assume that it is the one constructed in [Ste70, p. 167-170].*

We fix $0 < \eta < 1/4$, which is arbitrary but will be kept fixed in what follows. We denote by Q_i^* the cube with the same center as Q_i but expanded by the factor $(1+\eta)$, i.e., $Q_i^* = (1+\eta)Q_i$.

In [Ste70] the results in the two next propositions were also proved.

Proposition 3.4.6. *(i) Each point of $\mathbb{R}^n \setminus \mathbf{F}_{\varepsilon k}$ is contained in at most N of the cubes Q_i^*, $Q_i \in \mathcal{Q}$, where N is the same as in Definition 3.4.4(vi).*

(ii) If Q_χ intersects Q_i^*, then Q_χ and Q_i touch.

Proposition 3.4.7. *There exists a numerable collection of functions $\{\varphi_i\}_i \subset C_0^\infty(\mathbb{R}^n)$ such that*

(i) $\varphi_i(x) = 0, \quad$ *if* $x \notin Q_i^*$,

(ii) $\sum_i \varphi_i(x) = 1,\quad$ *if* $x \in \mathbb{R}^n \setminus \mathbf{F}_{\varepsilon k}$,

(iii) $|D^\alpha \varphi_i(x)| \leq A_\alpha (\operatorname{diam} Q_i)^{-|\alpha|},\quad x \in \mathbb{R}^n,\quad \alpha \in {\mathbb{N}_0}^n$.

In what follows, given a cube Q_i we will denote its center by x_i, its side length by s_i and its diameter by l_i.

We introduce some more notation:

$$C_i := (\mu_{\varepsilon k}(B(x_i, 6l_i)))^{-1}$$

The property presented in the next proposition was mentioned in [Jon94, p. 362].

Let $Q_i \in \mathcal{Q}$. Then it can be easily verified that there is $p_i \in \mathbf{F}_{\varepsilon k}$ such that $|p_i - x_i| \leq 5l_i$.

Proposition 3.4.8. *Let $p_i \in \mathbf{F}_{\varepsilon k}$ be such that $|p_i - x_i| \leq 5l_i$.*
If $|t - x_i| \leq 30l_i$, then

$$\mu_{\varepsilon k}(B(x_i, 6l_i)) \geq \mu_{\varepsilon k}(B(p_i, l_i)) \geq c\mu_{\varepsilon k}(B(p_i, 36l_i)) \geq c\mu_{\varepsilon k}(B(t, l_i)) \geq \mu_{\varepsilon k}(B(t, s_i)). \tag{3.4.14}$$

Definition 3.4.9. *Let*

$$I := \{i : s_i \leq 1\}. \tag{3.4.15}$$

We define, for $\widetilde{u} \in \widetilde{B}_p^{T_k(\sigma_\varepsilon), N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})$,

$$E_{\varepsilon,k}\widetilde{u}(x) := \sum_{i \in I} \varphi_i(x) C_i \int\limits_{|t - x_i| \leq 6l_i} \widetilde{u}(t) d\mu_{\varepsilon k}(t), \quad x \in \mathbb{R}^n \setminus \mathbf{F}_{\varepsilon k}. \tag{3.4.16}$$

Remark 3.4.10. *By the conditions on the support of $\widetilde{u}$, we can replace in the integral in* (3.4.16)

$$\int\limits_{|t - x_i| \leq 6l_i} \quad \text{by} \quad \int\limits_{t \in \Gamma_{\varepsilon k},\, |t - x_i| \leq 6l_i}$$

As we have already mentioned the construction and study of these extension operators follows [JW84] and [Jon94]. Next we present a collection of results which can be proved analogously to what was done in these works. In all of them everything was adapted to the sets and measures that we are considering and in order to guarantee that the constants involved are independent of k, $k \in \mathbb{N}_0$.

As far as the results stated in the next proposition are concerned, we refer to [JW84, p. 110] and [Jon94, p. 360].

Proposition 3.4.11. *Let $Q_i, Q_\chi \in \mathcal{Q}$ and $\{\varphi_i\}_i$ be as in Proposition 3.4.7. Assume that Q_i and Q_χ touch. Then*

$$|t - x_\chi| \leq 6l_\chi \quad \Rightarrow \quad |t - x_i| \leq 30l_i. \tag{3.4.17}$$

Moreover, there is $c > 0$ such that

$$C_i \leq c \cdot C_\chi \tag{3.4.18}$$

and, for all $\alpha \in {\mathbb{N}_0}^n$ there is $c_\alpha > 0$ such that

$$|D^\alpha \varphi_i(x)| \leq c_\alpha l_\chi^{-|\alpha|}, \quad x \in \mathbb{R}^n. \tag{3.4.19}$$

The properties referred in the next proposition, which can be easily verified, were also mentioned in [JW84, p. 110] and [Jon94, p. 360].

Proposition 3.4.12.

(i) If there is a number i such that $x \in Q_i$ with $s_i > 4$, then $E_{\varepsilon,k}\widetilde{u}(x) = 0$.

(ii) If there is a number j such that $x \in Q_j$ with $s_j \leq \frac{1}{4}$, then $\sum_{i \in I} \varphi_i(x) = \sum_i \varphi_i(x)$, where I is as in (3.4.15)*.*

We will denote by I_ν the set of all i such that $s_i = 2^{-\nu}$ and

$$\Delta_\nu := \bigcup_{i \in I_\nu} Q_i. \tag{3.4.20}$$

The property we present next was proved in [JW84, p. 110].

Proposition 3.4.13. *If $x \in \Delta_\nu$ and $|x-y| \leq \frac{2^{-\nu}}{2}$ then $y \in \Delta'_\nu := \bigcup_{i=\nu-2}^{\nu+3} \Delta_i$.*

We define, for $1 \leq p < \infty$

$$J_p(x_\tau, x_\chi) := \left(C_\chi C_\tau \int\limits_{\substack{t\in\Gamma_{\varepsilon k} \\ |t-x_\chi|\leq 30 l_\chi}} \int\limits_{\substack{r\in\Gamma_{\varepsilon k} \\ |r-x_\tau|\leq 30 l_\tau}} |\widetilde{u}(t)-\widetilde{u}(r)|^p d\mu_{\varepsilon k}(r) d\mu_{\varepsilon k}(t) \right)^{\frac{1}{p}} \tag{3.4.21}$$

and

$$J_\infty(x_\tau, x_\chi) := \sup_{\substack{t,r\in\Gamma_{\varepsilon k} \\ |t-x_\chi|\leq 30 l_\chi, |r-x_\tau|\leq 30 l_\tau}} |\widetilde{u}(t)-\widetilde{u}(r)| \tag{3.4.22}$$

The proof of the next lemma is analogous to the proof of Lemma D in [JW84, p. 111-112].

Lemma 3.4.14. *Let $1 \leq p \leq \infty$, $x \in Q_\chi$ and $y \in Q_\tau$.*

(i) If $s_\chi, s_\tau \leq \frac{1}{4}$, then

(a) $|E_{\varepsilon,k}\widetilde{u}(x) - E_{\varepsilon,k}\widetilde{u}(y)| \leq c J_p(x_\chi, x_\tau))$;

(b) $|D^\alpha(E_{\varepsilon,k}\widetilde{u})(x)| \leq c l_\chi^{-|\alpha|} J_p(x_\chi, x_\tau))$;

(c) Let b be an arbitrary number. If $1 \leq p < \infty$, then

$$|E_{\varepsilon,k}\widetilde{u}(x) - b| \leq c \left(C_\chi \int\limits_{\substack{t\in\Gamma_{\varepsilon k} \\ |t-x_\chi|\leq 30 l_\chi}} |\widetilde{u}(t) - b|^p d\mu_{\varepsilon k}(t) \right)^{\frac{1}{p}}. \tag{3.4.23}$$

For $p = \infty$,

$$|E_{\varepsilon,k}\widetilde{u}(x) - b| \leq c \sup_{\substack{t\in\Gamma_{\varepsilon k} \\ |t-x_\chi|\leq 30 l_\chi}} |\widetilde{u}(t) - b|. \tag{3.4.24}$$

(ii) For all $\alpha \in {\mathbb{N}_0}^n$,

$$|D^\alpha(E_{\varepsilon,k}\widetilde{u})(x)| \leq c l_\chi^{-|\alpha|} \left(C_\chi \int\limits_{\substack{t\in\Gamma_{\varepsilon k} \\ |t-x_\chi|\leq 30 l_\chi}} |\widetilde{u}(t)|^p d\mu_{\varepsilon k}(t) \right)^{\frac{1}{p}}, \tag{3.4.25}$$

if $1 \leq p < \infty$, and

$$|D^\alpha(E_{\varepsilon,k}\widetilde{u})(x)| \leq c \sup_{\substack{t\in\Gamma_{\varepsilon k} \\ |t-x_\chi|\leq 30 l_\chi}} |\widetilde{u}(t)|. \tag{3.4.26}$$

The proof of the next lemma is analogous to the proof of Lemma 2 in [JW84, p. 112-113].

Lemma 3.4.15. *Let* $b > 0$ *and* v *be a non-negative function defined on* $\mathbf{F}_{\varepsilon k}$. *Let*

$$g(x) := \int\limits_{|t-x_\chi|\le bl_\chi} v(t) d\mu_{\varepsilon k}(t), \quad x \in int Q_\chi, \quad \chi \in I_\nu. \tag{3.4.27}$$

Then, for $x_0 \in \mathbb{R}^n$ *and* $0 < r \le \infty$,

$$\int\limits_{\substack{x\in\Delta_\nu \\ |x-x_0|\le r}} g(x)dx \le c2^{-\nu n} \int\limits_{|t-x_0|\le r+d_0 2^{-\nu}} v(t) d\mu_{\varepsilon k}(t). \tag{3.4.28}$$

In particular, for $r = \infty$,

$$\int\limits_{x\in\Delta_\nu} g(x)dx \le c2^{-\nu n} \int_{\mathbf{F}_{\varepsilon k}} v(t) d\mu_{\varepsilon k}(t). \tag{3.4.29}$$

Here c *and* d_0 *depend only on* b *and* n. *In particular* $d_0 = (1/2 + b)\sqrt{n}$.
If additionally we assume that $\operatorname{supp} v \subset \Gamma_{\varepsilon k}$, *then in the second member in* (3.4.27)-(3.4.29) *we can add* $t \in \Gamma_{\varepsilon k}$ *as a condition for the elements in the integration region.*

In the next lemma we present a Hardy inequality which can be found in [JW84, p. 121, Lemma 3], where [Lei70] is referred for proofs.

Lemma 3.4.16. *Let* $0 < p < \infty$.

(i) For all $\alpha < 0$ *there is a positive number* c *such that*

$$\sum_{j=0}^{\infty} 2^{\alpha j} \Big(\sum_{i=0}^{j} \alpha_i\Big)^p \le c \sum_{j=0}^{\infty} 2^{\alpha j} \alpha_j^p, \quad \textit{for all} \quad (\alpha_j)_{j\in\mathbb{N}_0} \subset \mathbb{R}_0^+.$$

(ii) For all $\alpha > 0$ *there is a positive number* c *such that*

$$\sum_{j=0}^{\infty} 2^{\alpha j} \Big(\sum_{i=j}^{\infty} \alpha_i\Big)^p \le c \sum_{j=0}^{\infty} 2^{\alpha j} \alpha_j^p, \quad \textit{for all} \quad (\alpha_j)_{j\in\mathbb{N}_0} \subset \mathbb{R}_0^+.$$

Remark 3.4.17. *Let σ be an admissible sequence and Λ be an admissible function associated to σ, i.e., $\Lambda \in \mathcal{A}_\sigma$. Let a and A be given by*

$$a := \sigma \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}} \quad \text{and} \quad A(x) = \Lambda(x) h(x^{-1})^{\frac{1}{p}} x^{\frac{n}{p}}. \tag{3.4.30}$$

Then a is an admissible sequence and A is an admissible function associated to a.

Theorem 3.4.18. *Let $0 < \varepsilon \leq 1$ and $N_\varepsilon = (2^{\varepsilon j})_{j \in \mathbb{N}_0}$. Consider the conditions described in Assumption 3.4.2. Let $1 \leq p \leq \infty$ and σ be an admissible sequence such that $\underline{s}(\sigma) > 0$. Fix $R \in (0, \infty)$ and $\Lambda \in \mathcal{A}_\sigma$. Consider a and A as given by (3.4.30) and $M \in \mathbb{N}$ such that $\overline{s}(a) < M$. Let $\widetilde{u} \in \widetilde{B}_p^{T_k(\sigma_\varepsilon), N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})$. Then $E_{\varepsilon,k}\widetilde{u} \in L_p(\mathbb{R}^n)$ and there is $c > 0$ such that*

$$\begin{aligned} \|E_{\varepsilon,k}\widetilde{u} | B_p^{T_k(a_\varepsilon), N_\varepsilon}(\mathbb{R}^n)\|_M \;&:=\; a_{\varepsilon,k} \|E_{\varepsilon,k}\widetilde{u} | L_p(\mathbb{R}^n)\| \\ &+ \;\left(\int_{|r| \leq R} \left(A(2^{\varepsilon k}|r|^{-1}) \|\Delta_r^M (E_{\varepsilon,k}\widetilde{u}) | L_p(\mathbb{R}^n)\| \right)^p \frac{dr}{|r|^n} \right)^{\frac{1}{p}} \end{aligned}$$

can be estimated from above by

$$c h(2^{-\varepsilon k})^{\frac{1}{p}} 2^{\frac{n\varepsilon k}{p}} \|\widetilde{u} | \widetilde{B}_p^{T_k(\sigma_\varepsilon), N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\|, \tag{3.4.31}$$

where c is independent of k and γ^0 in (3.4.4). Furthermore,

$$(E_{\varepsilon,k}\widetilde{u})_{|\mathbf{F}_{\varepsilon k}} = \widetilde{u}. \tag{3.4.32}$$

Remark 3.4.19. *Let $g = E_{\varepsilon,k}\widetilde{u}$. By (3.4.32) we mean that for $\mu_{\varepsilon k}$-almost all $t_0 \in \mathbf{F}_{\varepsilon k}$*

$$\lim_{r \to 0} \frac{1}{|B(t_0, r)|} \int_{B(t_0,r)} g(x) dx = \widetilde{u}(t_0). \tag{3.4.33}$$

We can easily check that

$$B_p^{T_k(a_\varepsilon), N_\varepsilon}(\mathbb{R}^n) =_k B_p^{T_{[\varepsilon k]}(\sigma)(\boldsymbol{h}^*_{\varepsilon,k})^{\frac{1}{p}}(n)^{\frac{1}{p}}}(\mathbb{R}^n),$$

where we are writing $=_k$ to make clear that in this equality we mean equivalent norms with constants depending of k. In fact, for all $f \in B_p^{T_k(a_\varepsilon), N_\varepsilon}(\mathbb{R}^n)$,

$$\|f | B_p^{T_k(a_\varepsilon), N_\varepsilon}(\mathbb{R}^n)\| \sim 2^{\frac{n\varepsilon k}{p}} h(2^{-\varepsilon k})^{\frac{1}{p}} \|f | B_p^{T_{[\varepsilon k]}(\sigma)(\boldsymbol{h}^*_{\varepsilon,k})^{\frac{1}{p}}(n)^{\frac{1}{p}}}(\mathbb{R}^n)\|,$$

where the equivalence constants are now independent of k *and where we applied* (2.1.10). *So, we can adapt the arguments in the proof of Theorem 3.4.15, pp. 114-116, of [Bri01], to conclude that, in these conditions, for* $1 < p < \infty$,

$$\mathrm{tr}_{\mathbf{F}_{\varepsilon k}}(E_{\varepsilon,k}\widetilde{u}) = \widetilde{u}, \tag{3.4.34}$$

where $\mathrm{tr}_{\mathbf{F}_{\varepsilon k}}$ *is given by Definition 3.2.1. If* $p = \infty$, (3.4.34) *follows immediately from* (3.4.33), *because in this case the trace is the pointwise restriction.*

Proof. Step 1: We will estimate $\|E_{\varepsilon,k}\widetilde{u}|L_p(\mathbb{R}^n)\|$. Let $1 \leq p < \infty$. Then, by Lemma 3.4.14, (ii),

$$|(E_{\varepsilon,k}\widetilde{u})(x)|^p \leq c\,C_\chi \int\limits_{\substack{t\in\Gamma_{\varepsilon k}\\ |t-x_\chi|\leq 30 l_\chi}} |\widetilde{u}(t)|^p d\mu_{\varepsilon k}(t),$$

where χ is such that $x \in Q_\chi$. We recall that, by Proposition 3.4.12, (i), we can exclude from consideration all x which belong to a cube Q_i such that $s_i > 4$.
It follows from Proposition 3.4.8 that

$$|(E_{\varepsilon,k}\widetilde{u})(x)|^p \leq c \int\limits_{\substack{t\in\Gamma_{\varepsilon k}\\ |t-x_\chi|\leq 30 l_\chi}} \frac{|\widetilde{u}(t)|^p}{\mu_{\varepsilon k}(B(t,s_\chi))} d\mu_{\varepsilon k}(t). \tag{3.4.35}$$

So, applying (3.4.8), (3.4.35) and Lema 3.4.15, we obtain

$$\begin{aligned}
\|E_{\varepsilon,k}\widetilde{u}|L_p(\mathbb{R}^n)\|^p &= \sum_{\nu\geq -2}\int_{\Delta_\nu} |E_{\varepsilon,k}\widetilde{u}(x)|^p dx \\
&\leq \sum_{\nu\geq -2}\sum_{\chi\in I_\nu}\int_{Q_\chi}\int\limits_{\substack{t\in\Gamma_{\varepsilon k}\\ |t-x_\chi|\leq 30 l_\chi}} \frac{|\widetilde{u}(t)|^p}{\mu_{\varepsilon k}(B(t,s_\chi))} d\mu_{\varepsilon k}(t)dx \\
&\leq c\sum_{\nu\geq -2} h^*_{\varepsilon,k}(2^{-\nu})2^{-\nu n}\int_{\Gamma_{\varepsilon k}} |\widetilde{u}(t)|^p d\mu_{\varepsilon k}(t).
\end{aligned}$$

Choosing $\delta \in (0, -\overline{s}(\boldsymbol{h}))$, first by (3.4.2) and (3.4.7) and then by (3.4.1) and (3.4.12), we get to

$$\|E_{\varepsilon,k}\widetilde{u}|L_p(\mathbb{R}^n)\|^p \lesssim \|\widetilde{u}|L_p(\Gamma_{\varepsilon k},\mu_{\varepsilon k})\|^p \sum_{\nu\geq -2} 2^{-\nu(n-\overline{s}(\boldsymbol{h})-\delta)} \lesssim \sigma_{\varepsilon,k}^{-p}\,\|\widetilde{u}|\widetilde{B}_p^{T_k(\sigma_\varepsilon),N_\varepsilon}(\Gamma_{\varepsilon k},\mu_{\varepsilon k})\|^p$$

Therefore, as a is given by (3.4.30), $a_{\varepsilon,k} \,\|E_{\varepsilon,k}\widetilde{u}|L_p(\mathbb{R}^n)\|$ can be estimated by (3.4.31). With the usual modifications we can prove something similar for the case $p=\infty$.

Step 2: Let

$$R := \frac{2^{-11}}{2M} \qquad \text{and} \qquad r_m := \frac{2^{-m}}{2M} \tag{3.4.36}$$

Then

$$\begin{aligned}\int_{|r|\le R} &\big(A(2^{\varepsilon k}|r|^{-1})\|\Delta_r^M(E_{\varepsilon,k}\widetilde{u})|L_p(\mathbb{R}^n)\|\big)^p \frac{dr}{|r|^n}\\ \le \quad & c\sum_{m=11}^{\infty} A(2^{\varepsilon k}2^m)^p 2^{mn} \int\limits_{r_{m+1}\le|r|\le r_m} \int_{\mathbb{R}^n} |\Delta_r^M(E_{\varepsilon,k}\widetilde{u})(x)|^p dx dr\end{aligned} \tag{3.4.37}$$

Applying Propositions 3.4.12(i) and 3.4.13 one can easily prove that, if $x\in\Delta_\nu$, with $\nu\le -6$, and $|r|\le \frac{2^{-6}}{2M}$, then $\Delta_r^M(E_{\varepsilon,k}\widetilde{u})(x)=0$. Let

$$G_m := \bigcup_{\nu=-5}^{m} \Delta_\nu \qquad \text{and} \qquad F_{m+1} := \bigcup_{\nu=m+1}^{\infty} \Delta_\nu. \tag{3.4.38}$$

Then, (3.4.37) can be bounded from above by

$$\sum_{m=11}^{\infty} A_m + \sum_{m=11}^{\infty} B_m, \tag{3.4.39}$$

where

$$A_m = A(2^{\varepsilon k}2^m)^p 2^{mn} \int\limits_{r_{m+1}\le|r|\le r_m} \int_{G_m} |\Delta_r^M(E_{\varepsilon,k}\widetilde{u})(x)|^p dx dr \tag{3.4.40}$$

and

$$B_m = A(2^{\varepsilon k}2^m)^p 2^{mn} \int\limits_{r_{m+1}\le|r|\le r_m} \int_{F_{m+1}} |\Delta_r^M(E_{\varepsilon,k}\widetilde{u})(x)|^p dx dr \tag{3.4.41}$$

Let us first estimate $\sum A_m$ from above. We will apply the equality

$$\Delta_1^s g(0) = \int_0^1 \dots \int_0^1 g^{(s)}(\theta_1 + \dots + \theta_s) d\theta_1 \dots d\theta_s, \tag{3.4.42}$$

valid for any infinitely differentiable function g of one variable (cf. [JW84, p. 115]). So applying (3.4.42) to the function $g(t) := E_{\varepsilon,k}\widetilde{u}(x+tr)$, the Hölder inequality and a change of variables we get to

$$\int_{G_m} |\Delta_r^M(E_{\varepsilon,k}\widetilde{u})(x)|^p dx \lesssim \sum_{\nu=-5}^{m} |r|^{Mp} \sum_{|j|=M} \int_{\Delta'_\nu} |D^j(E_{\varepsilon,k}\widetilde{u})(x)|^p dx,$$

where Δ'_ν is as in Proposition 3.4.13 and so

$$\sum_{m=11}^{\infty} A_m \lesssim \sum_{m=11}^{\infty} A(2^{\varepsilon k}2^m)^p 2^{-mMp} \sum_{\nu=-7}^{m+3} \sum_{|j|=M} \int_{\Delta_\nu} |D^j(E_{\varepsilon,k}\widetilde{u})(x)|^p dx. \tag{3.4.43}$$

Let $\delta \in (0, M - \overline{s}(a))$. Using the fact that A is an admissible function associated to a and the properties of Boyd indices one can prove that

$$\frac{A(2^{\varepsilon k}2^m)}{A(2^{\varepsilon k}2^\nu)} \lesssim 2^{(\overline{s}(a)+\delta)(m-\nu)}, \quad m,\nu \in \mathbb{Z}, \quad m \geq 11, \quad -7 \leq \nu \leq m+3. \tag{3.4.44}$$

Then applying (3.4.43)-(3.4.44) and Lemma 3.4.16, we obtain

$$\begin{aligned}
\sum_{m=11}^{\infty} A_m &\lesssim \sum_{m=11}^{\infty} 2^{-mp(M-\overline{s}(a)-\delta)} \sum_{\nu=-7}^{m+3} 2^{-\nu(\overline{s}(a)+\delta)p} A(2^{\varepsilon k}2^\nu)^p \sum_{|j|=M} \int_{\Delta_\nu} |D^j(E_{\varepsilon,k}\widetilde{u})(x)|^p dx \\
&\lesssim \sum_{\nu=-7}^{\infty} 2^{-\nu Mp} A(2^{\varepsilon k}2^\nu)^p \sum_{|j|=M} \int_{\Delta_\nu} |D^j(E_{\varepsilon,k}\widetilde{u})(x)|^p dx \\
&=: \sum_{\nu=-7}^{\infty} D_\nu.
\end{aligned}$$

We consider $\nu \geq 2$. Applying Lemma 3.4.14(i)(b), Proposition 3.4.8 and Lemma 3.4.15, we conclude that there is $d_0 > 0$ such that

$$\int_{\Delta_\nu} |D^j(E_{\varepsilon,k}\widetilde{u})(x)|^p dx \lesssim 2^{\nu Mp} 2^{-\nu n} \iint\limits_{\substack{t,v\in\Gamma_{\varepsilon k} \\ |t-v|\leq d_0 2^{-\nu}}} \frac{|\widetilde{u}(t)-\widetilde{u}(v)|^p}{(\mu_{\varepsilon k}(B(t,2^{-\nu})))^2} d\mu_{\varepsilon k}(v) d\mu_{\varepsilon k}(t).$$

Therefore, as A is given by (3.4.30), we get to

$$\sum_{\nu=2}^{\infty} D_\nu \lesssim \sum_{\nu=2}^{\infty}\sum_{j=\nu}^{\infty} \Lambda(2^{\varepsilon k}2^\nu)^p h(2^{-\varepsilon k}2^{-\nu})\, 2^{\varepsilon k n} \iint\limits_{\substack{t,v\in\Gamma_{\varepsilon k} \\ d_0 2^{-(j+1)}\leq|t-v|\leq d_0 2^{-j}}} \frac{|\widetilde{u}(t)-\widetilde{u}(v)|^p}{(\mu_{\varepsilon k}(B(t,2^{-\nu})))^2} d\mu_{\varepsilon k}(v) d\mu_{\varepsilon k}(t) \tag{3.4.45}$$

We fix now $0 < \delta' < \min\{\underline{s}(\sigma), -\overline{s}(\boldsymbol{h})\}$. Applying the fact that $\Lambda \in \mathcal{A}_\sigma$ one can prove that

$$\frac{\Lambda(2^{\varepsilon k}2^{\nu})}{\Lambda(2^{\varepsilon k}2^{j})} \lesssim 2^{-(\underline{s}(\sigma)-\delta')(j-\nu)}, \quad j,\nu \in \mathbb{N}, \quad , j \geq \nu \geq 2. \tag{3.4.46}$$

Applying (3.4.2), (3.4.7) and (3.4.8), we get to

$$\frac{h(2^{-\varepsilon k}2^{-\nu})}{(\mu_{\varepsilon k}(t,2^{-\nu}))^2} \lesssim \frac{h(2^{-\varepsilon k})}{h^*_{\varepsilon,k}(2^{-j})}\, 2^{(j-\nu)(\overline{s}(\boldsymbol{h})+\delta')}, \quad j,\nu \in \mathbb{N}, \quad , j \geq \nu \geq 2, \quad t \in \Gamma_{\varepsilon k}. \tag{3.4.47}$$

Therefore considering

$$\alpha := (\underline{s}(\sigma) - \delta')p - \overline{s}(\boldsymbol{h}) - \delta',$$

which is a positive number, it follows from (3.4.45)-(3.4.47) that

$$\begin{aligned}\sum_{\nu=2}^{\infty} D_\nu \;&\lesssim\; h(2^{-\varepsilon k})\, 2^{\varepsilon k n} \sum_{\nu=2}^{\infty}\sum_{j=\nu}^{\infty} 2^{-j\alpha}2^{\nu\alpha} \int\!\!\int_{\substack{t,v\in\Gamma_{\varepsilon k}\\ d_0 2^{-(j+1)}\leq|t-v|\leq d_0 2^{-j}}} \frac{\Lambda(2^{\varepsilon k}|t-v|^{-1})^p}{h^*_{\varepsilon,k}(|t-v|)} |\widetilde{u}(t)-\widetilde{u}(v)|^p d\mu_{\varepsilon k}(v) d\mu_{\varepsilon k}(t)\\
&\lesssim h(2^{-\varepsilon k})\, 2^{\varepsilon k n} \sum_{j=2}^{\infty} 2^{-j\alpha} \left(\int\!\!\int_{\substack{t,v\in\Gamma_{\varepsilon k}\\ d_0 2^{-(j+1)}\leq|t-v|\leq d_0 2^{-j}}} \frac{\Lambda(2^{\varepsilon k}|t-v|^{-1})^p}{h^*_{\varepsilon,k}(|t-v|)} |\widetilde{u}(t)-\widetilde{u}(v)|^p d\mu_{\varepsilon k}(v) d\mu_{\varepsilon k}(t) \right) \sum_{\nu\leq j} 2^{\nu\alpha}\end{aligned}$$

and hence, recalling (3.4.12), we proved that

$$\sum_{\nu=2}^{\infty} D_\nu \lesssim h(2^{-\varepsilon k})\, 2^{\varepsilon k n} \|\widetilde{u}|\widetilde{B}_p^{T_k(\sigma_\varepsilon),N_\varepsilon}(\Gamma_{\varepsilon k},\mu_{\varepsilon k})\|^p. \tag{3.4.48}$$

An analogous estimate can be proved for $\sum_{\nu=-7}^{1} D_\nu$, applying Lemma 3.4.14(ii), instead of Lemma 3.4.14(i)(b).
Now let us study $\sum_{m=11}^{\infty} B_m$.

Applying (2.4.1), the property referred in Definition 3.4.4(iv) and Proposition 3.4.13, we can prove that there exist $c, c' > 0$ such that, for $|r| < r_m$,

$$\int_{F_{m+1}} |\Delta_r^M(E_{\varepsilon,k}\widetilde{u})(x)|^p dx \leq c \int_{F_{m-4}} |(E_{\varepsilon,k}\widetilde{u})(x) - (E_{\varepsilon,k}\widetilde{u})(x+r)|^p dx$$

and

$$\int_{F_{m-4}} |(E_{\varepsilon,k}\widetilde{u})(x) - (E_{\varepsilon,k}\widetilde{u})(x+r)|^p dx \leq c'|r|^{-n} \iint_{\substack{|t|<2|r|\\ x,x+t\in F_{m-9}}} |(E_{\varepsilon,k}\widetilde{u})(x) - (E_{\varepsilon,k}\widetilde{u})(x+t)|^p dx dt.$$

Applying these inequalities we obtain

$$\sum_{m=11}^{\infty} B_m \lesssim \sum_{m=11}^{\infty} A(2^{\varepsilon k}2^m)^p 2^{mn} \iint\limits_{\substack{x,y\in F_{m-9}\\ |x-y|\le 2r_m}} |(E_{\varepsilon,k}\widetilde{u})(x)-(E_{\varepsilon,k}\widetilde{u})(y)|^p dxdy \tag{3.4.49}$$

Applying Lemma 3.4.14(i)(a), Proposition 3.4.8 and then (twice) Lemma 3.4.15, we can conclude that there exist $c, d_0' > 0$ such that, for $m \geq 11$ and $\nu, \beta \geq m-9$,

$$\iint\limits_{\substack{x\in\Delta_\nu,\, y\in\Delta_\beta\\ |x-y|\le 2r_m}} |(E_{\varepsilon,k}\widetilde{u})(x)-(E_{\varepsilon,k}\widetilde{u})(y)|^p dxdy \leq c2^{-\nu n}2^{-\beta n} \iint\limits_{\substack{t,v\in\Gamma_{\varepsilon k}\\ |t-v|\le d_0' 2^{-m}}} \frac{|\widetilde{u}(t)-\widetilde{u}(v)|^p}{h^*_{\varepsilon,k}(2^{-\nu})h^*_{\varepsilon,k}(2^{-\beta})} d\mu_{\varepsilon k}(v)d\mu_{\varepsilon k}(t). \tag{3.4.50}$$

Let $0 < \delta'' < n + \underline{s}(\boldsymbol{h})$. By (3.4.10), for all $\nu \geq m-9$,

$$\frac{h^*_{\varepsilon,k}(2^{-m})}{h^*_{\varepsilon,k}(2^{-\nu})} \lesssim 2^{(\nu-m)(-\underline{s}(\boldsymbol{h})+\delta'')}. \tag{3.4.51}$$

Hence by (3.4.49)-(3.4.51)

$$\sum_{m=11}^{\infty} B_m \lesssim \sum_{m=11}^{\infty} \frac{A(2^{\varepsilon k}2^m)^p 2^{-mn}}{(h^*_{\varepsilon,k}(2^{-m}))^2} \iint\limits_{\substack{t,v\in\Gamma_{\varepsilon k}\\ |t-v|\le d_0' 2^{-m}}} |\widetilde{u}(t)-\widetilde{u}(v)|^p d\mu_{\varepsilon k}(v)d\mu_{\varepsilon k}(t). \tag{3.4.52}$$

Following the techniques used to study $\sum_{\nu=2}^{\infty} D_\nu$ we obtain for $\sum_{m=11}^{\infty} B_m$ an estimation similar to the one in (3.4.48). Analogously one can study the case $p = \infty$, concluding the proof of the first part of the theorem.

Step 3: Let us prove (3.4.32) according to the interpretation in (3.4.33).

Let

$$\mathcal{C}(\widetilde{u},\varepsilon,k,h,t_0) := \int_{\Gamma_{\varepsilon k}} \frac{\Lambda(2^{\varepsilon k}|t-t_0|^{-1})^p}{h^*_{\varepsilon,k}(|t-t_0|)} |\widetilde{u}(t)-\widetilde{u}(t_0)|^p d\mu_{\varepsilon k}(t) \tag{3.4.53}$$

As the expression in (3.4.12) is finite, then, for $\mu_{\varepsilon k}$-almost all $t_0 \in \Gamma_{\varepsilon k}$, $\mathcal{C}(f,\varepsilon,k,h,t_0)$ is finite. If $t_0 \in \mathbf{F}_{\varepsilon k} \setminus \Gamma_{\varepsilon k}$ then, recalling (3.4.5)-(3.4.6) and the conditions on the support of $\widetilde{u}$ given in (3.4.11), we conclude that the expression in (3.4.53) can be estimated from above by

$$\begin{aligned} \int\limits_{\substack{\Gamma_{\varepsilon k}\\ c_0\le |t-t_0|\le \operatorname{diam}\mathbf{F}_{\varepsilon k}}} \frac{\Lambda(2^{\varepsilon k}|t-t_0|^{-1})^p}{h^*_{\varepsilon,k}(|t-t_0|)} |\widetilde{u}(t)|^p d\mu_{\varepsilon k}(t) &\leq \|\widetilde{u}|L_p(\mathbf{F}_{\varepsilon k},\mu_{\varepsilon k})\|^p \sup_{c_0\le x\le \operatorname{diam}\mathbf{F}_{\varepsilon k}} \frac{\Lambda(2^{\varepsilon k}x^{-1})^p}{h^*_{\varepsilon,k}(x)} \\ &\leq c_k \|\widetilde{u}|L_p(\mathbf{F}_{\varepsilon k},\mu_{\varepsilon k})\|^p, \end{aligned} \tag{3.4.54}$$

where we used the fact that both Λ and $h^*_{\varepsilon,k}$ are continuous functions and where c_k is a positive number depending on k. So, for $\mu_{\varepsilon k}$-almost all $t_0 \in \mathbf{F}_{\varepsilon k}$, $\mathcal{C}(f, \varepsilon, k, h, t_0)$ is finite. We consider any such t_0. Let $x \in \mathbb{R}^n \setminus \mathbf{F}_{\varepsilon k}$ be such that $|x - t_0| < r$. As we will consider $r \to 0$ we may consider r conveniently small so that $x \in Q_\chi$ with $s_\chi = 2^{-\nu} \leq 1/4$ (we refer to Definition 3.4.4(iii)).

By Lemma 3.4.14, Proposition 3.4.8 and Lemma 3.4.15, there is $d_0 > 0$ such that

$$\int\limits_{\substack{x\in\Delta_\nu \\ |x-t_0|\leq r}} |E_{\varepsilon,k}\widetilde{u}(x) - \widetilde{u}(t_0)|^p dx \leq c2^{-\nu n} \int\limits_{\substack{t\in\Gamma_{\varepsilon k} \\ |t-t_0|\leq r+d_0 2^{-\nu}}} \frac{|\widetilde{u}(t) - \widetilde{u}(t_0)|^p}{\mu_{\varepsilon k}(B(t, 2^{-\nu}))} \, d\mu_{\varepsilon k}(t).$$

Let $\tau = \tau(r)$ be an integer such that

$$2^{-\tau} \leq \frac{r}{\sqrt{n}} \leq 2^{-(\tau-1)}.$$

Then, also applying (3.4.8) and (3.4.10),

$$\begin{aligned}
\int_{|x-t_0|\leq r} |E_{\varepsilon,k}\widetilde{u}(x) - \widetilde{u}(t_0)|^p dx \; &\lesssim \; \sum_{\nu=\tau}^{\infty} 2^{-\nu n} \int\limits_{\substack{t\in\Gamma_{\varepsilon k} \\ |t-t_0|\leq r+d_0 2^{-\nu}}} \frac{|\widetilde{u}(t) - \widetilde{u}(t_0)|^p}{\mu_{\varepsilon k}(B(t, 2^{-\nu}))} \, d\mu_{\varepsilon k}(t) \\
&\lesssim \; \frac{2^{-\tau n}}{h^*_{\varepsilon,k}(2^{-\tau})} \int\limits_{\substack{t\in\Gamma_{\varepsilon k} \\ |t-t_0|\leq d_0'' 2^{-\tau}}} |\widetilde{u}(t) - \widetilde{u}(t_0)|^p d\mu_{\varepsilon k}(t).
\end{aligned}$$

Therefore, considering $0 < \delta' < \min\{\underline{s}(\sigma), -\overline{s}(\boldsymbol{h})\}$,

$$\begin{aligned}
&\frac{1}{|B(t_0, r)|} \int_{|x-t_0|\leq r} |E_{\varepsilon,k}\widetilde{u}(x) - \widetilde{u}(t_0)|^p dx \\
&\quad\lesssim \frac{1}{h^*_{\varepsilon,k}(2^{-\tau})} \int\limits_{\substack{t\in\Gamma_{\varepsilon k} \\ |t-t_0|\leq d_0' 2^{-\tau}}} |\widetilde{u}(t) - \widetilde{u}(t_0)|^p d\mu_{\varepsilon k}(t) \\
&\quad\lesssim \sum_{j=\tau}^{\infty} 2^{(j-\tau)(\overline{s}(\boldsymbol{h})+\delta')} 2^{-(\underline{s}(\sigma)-\delta')jp} \int\limits_{\substack{t\in\Gamma_{\varepsilon k} \\ d_0' 2^{-(j+1)}\leq|t-t_0|\leq d_0' 2^{-j}}} \frac{\Lambda(2^{\varepsilon k} 2^j)^p}{h^*_{\varepsilon,k}(2^{-j})} \, |\widetilde{u}(t) - \widetilde{u}(t_0)|^p d\mu_{\varepsilon k}(t) \\
&\quad\lesssim \mathcal{C}(\widetilde{u}, \varepsilon, k, h, t_0) 2^{\tau(-\overline{s}(\boldsymbol{h})-\delta')} \sum_{j=\tau}^{\infty} 2^{-j(-\overline{s}(\boldsymbol{h})-\delta'+(\underline{s}(\sigma)-\delta')p)} \\
&\quad\lesssim \mathcal{C}(\widetilde{u}, \varepsilon, k, h, t_0) 2^{-\tau(\underline{s}(\sigma)-\delta')p},
\end{aligned}$$

which converges to 0 as $\tau \to \infty$, i.e., as $r \to 0$, concluding the proof. □

3.5 Non-smooth atomic decompositions

Next, we present the definition of what we call non-smooth atoms (on a compact set). As we shall see later, it is convenient for us to consider two kinds of non-smooth atoms. Recall the notation introduced in Assumption 3.3.3.

Definition 3.5.1. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set, $0 < \varepsilon \leq 1$ and $d > c_{\varepsilon,2}$, where $c_{\varepsilon,2}$ is as in* (3.3.3)*. Let $0 < p \leq \infty$ and σ be an admissible sequence. Then a Lipschitz-continuous function $a_\Gamma^{j,m}$ on Γ is called a d-$(\sigma,p)^*_\Gamma$-ε-atom if for $j \in \mathbb{N}_0$, $m = 1, ..., M_j$,*

(a) $\operatorname{supp} a_\Gamma^{j,m} \subset B(\gamma^{j,m}, d2^{-\varepsilon j}) \cap \Gamma$,

(b) $|a_\Gamma^{j,m}(\gamma)| \leq \sigma_j^{-1}\, h(2^{-\varepsilon j})^{-\frac{1}{p}}, \qquad \gamma \in \Gamma,$

(c) $|a_\Gamma^{j,m}(\gamma) - a_\Gamma^{j,m}(\delta)| \leq \sigma_j^{-1}\, h(2^{-\varepsilon j})^{-\frac{1}{p}} 2^{\varepsilon j} |\gamma - \delta|, \qquad \gamma, \delta \in \Gamma.$

Definition 3.5.2. *Let Γ, ε, p and σ be as in Definition 3.5.1. Let $d > c_{1,2}$ where $c_{1,2}$ is as in* (3.3.3) *for the 2^{-j}-approximate lattices $\{\delta^{j,t}\}_t$ (with $\varepsilon = 1$). A Lipschitz-continuous function $a_\Gamma^{j,t}$ on Γ is called a d-$(\sigma,p,\varepsilon)^{**}_\Gamma$-atom if for $j \in \mathbb{N}_0$, $t = 1, ..., T_j$,*

(a) $\operatorname{supp} a_\Gamma^{j,t} \subset B(\delta^{j,t}, d2^{-j}) \cap \Gamma$,

(b) $|a_\Gamma^{j,t}(\gamma)| \leq \sigma_j^{-1}\, h(2^{-j})^{-\frac{1}{p}}, \qquad \gamma \in \Gamma,$

(c) $|a_\Gamma^{j,t}(\gamma) - a_\Gamma^{j,t}(\delta)| \leq \sigma_j^{-1}\, h(2^{-j})^{-\frac{1}{p}} 2^{\varepsilon j} |\gamma - \delta|^\varepsilon, \qquad \gamma, \delta \in \Gamma.$

Theorem 3.5.3. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set. Let $0 < \varepsilon \leq 1$, $1 < p \leq \infty$ and β be an admissible sequence. Suppose that*

$$-n < \underline{s}(\boldsymbol{h}) \leq \overline{s}(\boldsymbol{h}) < 0.$$

(i) *Let $d > c_{\varepsilon,2}$ and*

$$0 < \underline{s}(\beta) \leq \overline{s}(\beta) < 1.$$

Then $\mathbb{B}_p^\beta(\Gamma)$ *is the collection of all* $f \in L_p(\Gamma)$ *such that*

$$f = \sum_{k=0}^{\infty} \sum_{m=1}^{M_k} \nu_{k,m} a_\Gamma^{k,m}(\gamma), \qquad \text{in } L_p(\Gamma), \tag{3.5.1}$$

for some $\nu \in b_p^\Gamma$ *and some family of* d*-*$(\beta_\varepsilon, p)_\Gamma^*$*-*$\varepsilon$*-atoms,* $a_\Gamma^{k,m}$*. Furthermore,*

$$\|f|\mathbb{B}_p^\beta(\Gamma)\| \sim \inf \|\nu|b_p^\Gamma\|,$$

where the infimum is taken over all representations (3.5.1)*.*

(ii) *Let* $d > c_{1,2}$ *and*

$$0 < \underline{s}(\beta) \le \overline{s}(\beta) < \varepsilon.$$

Then a corresponding result is true taking d*-*$(\beta, p, \varepsilon)_\Gamma^{**}$*-atoms (with* T_k *instead of* M_k *in* (3.5.1)*).*

Proof. Step 1. We will only present the proof for d-$(\beta_\varepsilon, p)_\Gamma^*$-$\varepsilon$-atoms, because the proof for d-$(\beta, p, \varepsilon)_\Gamma^{**}$-atoms is analogous. It follows immediately that all $f \in \mathbb{B}_p^\sigma(\Gamma)$ admits a representation (3.5.1) with $\|\nu|b_p^\Gamma\| \lesssim \|f|\mathbb{B}_p^\beta(\Gamma)\|$, because the restriction to Γ of d-$(\beta_\varepsilon, p)_K$-ε-atoms are special d-$(\beta_\varepsilon, p)_\Gamma^*$-$\varepsilon$-atoms.

Step 2. Let $f \in L_p(\Gamma)$ be as in (3.5.1), with $\nu \in b_p^\Gamma$. Let us prove that $f \in \mathbb{B}_p^\beta(\Gamma)$ and

$$\|f|\mathbb{B}_p^\beta(\Gamma)\| \lesssim \|\nu|b_p^\Gamma\|.$$

Let $k \in \mathbb{N}_0$, $m \in \{1, ..., M_k\}$ and $a_\Gamma^{k,m}$ be a d-$(\beta_\varepsilon, p)_\Gamma^*$-$\varepsilon$-atom located in $B(\gamma^{k,m}, d2^{-\varepsilon k})$, according to Definition 3.5.1. Following the notation used in Assumption 3.4.2, consider $\gamma^0 = \gamma^{k,m}$ and $c_0 = d$. Recall that

$$\mathbf{F}_{\varepsilon k} = D_{\varepsilon k}\Gamma \quad \text{and} \quad \Gamma_{\varepsilon k} = D_{\varepsilon k}(B(\gamma^{k,m}, 2d2^{-\varepsilon k}) \cap \Gamma) = \mathbf{F}_{\varepsilon k} \cap B(2^{\varepsilon k}\gamma^{k,m}, 2d).$$

Let $\alpha \in (0, 1 - \overline{s}(\beta))$, $\sigma = (\alpha)\beta$ and Λ be an admissible function associated to σ. Then

$$0 < \alpha + \underline{s}(\beta) = \underline{s}(\sigma) \le \overline{s}(\sigma) = \alpha + \overline{s}(\beta) < 1$$

To simplify the notation, let us temporarily denote by $\widetilde{u}$ the function $a_\Gamma^{k,m}(2^{-\varepsilon k}\cdot)$. Then

$$\widetilde{u} \in L_p(\mathbf{F}_{\varepsilon k}, \mu_{\varepsilon k}), \quad \text{and} \quad \operatorname{supp} \widetilde{u} \subset D_{\varepsilon k} B^\Gamma(d2^{-\varepsilon k}).$$

Consider $N_\varepsilon = (2^{\varepsilon j})_{j\in\mathbb{N}_0}$. Let us prove that

$$\|\widetilde{u}|\widetilde{B}_p^{T_k(\sigma_\varepsilon),N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| \leq ch(2^{-\varepsilon k})^{-\frac{1}{p}},$$

where c in independent of k. We present the proof for $1 < p < \infty$. The proof for $p = \infty$ is analogous, with the usual modifications. We can easily see that

$$\sigma_{\varepsilon,k}\|\widetilde{u}|L_p(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})\| \leq c_4^{\frac{1}{p}} h(2^{-\varepsilon k})^{-\frac{1}{p}}, \tag{3.5.2}$$

where c_4 is as in (3.4.9). Let $\delta \in (0, 1 - \overline{s}(\sigma))$. Then

$$\begin{aligned}
&\int_{\Gamma_{\varepsilon k}}\int_{\Gamma_{\varepsilon k}} \frac{\Lambda(2^{\varepsilon k}|t-v|^{-1})^p}{h^*_{\varepsilon,k}(|t-v|)}|\widetilde{u}(t) - \widetilde{u}(v)|^p d\mu_{\varepsilon k}(t) d\mu_{\varepsilon k}(v) \\
&\quad\leq \beta_{\varepsilon,k}^{-p}\, h(2^{-\varepsilon k})^{-1} \int_{\Gamma_{\varepsilon k}}\int_{\Gamma_{\varepsilon k}} \frac{\Lambda(2^{\varepsilon k}|t-v|^{-1})^p}{h^*_{\varepsilon,k}(|t-v|)}|t-v|^p d\mu_{\varepsilon k}(t) d\mu_{\varepsilon k}(v) \\
&\quad\lesssim \beta_{\varepsilon,k}^{-p}\, h(2^{-\varepsilon k})^{-1} \int_{t\in\Gamma_{\varepsilon k}} \sum_{j=0}^{+\infty} \Lambda(2^{\varepsilon(k+j)})^p 2^{-\varepsilon jp} d\mu_{\varepsilon k}(t) \\
&\quad\lesssim 2^{\varepsilon\alpha kp} h(2^{-\varepsilon k})^{-1} \mu_{\varepsilon k}(\Gamma_{\varepsilon k}) \sum_{j=0}^{+\infty} \big(\frac{\sigma_{\varepsilon,k+j}}{\sigma_{\varepsilon,k}}\big)^p 2^{-\varepsilon jp} \\
&\quad\lesssim 2^{\varepsilon\alpha kp} h(2^{-\varepsilon k})^{-1} \sum_{j=0}^{+\infty} 2^{-\varepsilon jp(1-(\overline{s}(\sigma)+\delta))} \\
&\quad\lesssim 2^{\varepsilon\alpha kp} h(2^{-\varepsilon k})^{-1}
\end{aligned}$$

Then $\widetilde{u} \in \widetilde{B}_p^{T_k(\sigma_\varepsilon),N_\varepsilon}(\Gamma_{\varepsilon k}, \mu_{\varepsilon k})$. By Theorems 3.4.18 and 2.4.1 and (3.5.2), we can conclude that $E_{\varepsilon,k}\widetilde{u} \in B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)$ and

$$a = \sigma \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}} = (\alpha)\beta \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}},$$

and

$$\|E_{\varepsilon,k}\widetilde{u}|B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \lesssim 2^{\varepsilon\alpha k} 2^{\frac{\varepsilon kn}{p}}.$$

We consider a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\varphi(x) = 1 \quad \text{if} \quad |x| \leq d \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \geq 2d.$$

Recalling that $\widetilde{u}$ denotes the function $a_\Gamma^{k,m}(2^{-\varepsilon k}\cdot)$, let

$$a^{k,m} := \varphi\big(2^{\varepsilon k}(\cdot - \gamma^{k,m})\big) \cdot E_{\varepsilon,k}\big(a_\Gamma^{k,m}(2^{-\varepsilon k}\cdot)\big).$$

Then

$$\|a^{k,m}|B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \lesssim 2^{\varepsilon\alpha k}2^{\frac{\varepsilon kn}{p}}, \qquad \operatorname{supp} a^{k,m} \subset B(2^{\varepsilon k}\gamma^{k,m}, 2d) \tag{3.5.3}$$

and

$$\operatorname{tr}_{\mathbf{F}_{\varepsilon k}}(a_\Gamma^{k,m}(2^{-\varepsilon k}\cdot)) = a_\Gamma^{k,m}(2^{-\varepsilon k}\cdot). \tag{3.5.4}$$

We refer to Remark 3.4.19 for some comments about (3.5.4).

Let

$$b^{k,m} := a^{k,m}(2^{\varepsilon k}\cdot).$$

By Remark 2.4.8, $b^{k,m} \in B_p^{a_\varepsilon,N_\varepsilon}(\mathbb{R}^n)$ and so, applying Proposition 2.1.7 and Theorem 2.4.7, we obtain

$$\|b^{k,m}|B_p^a(\mathbb{R}^n)\| \sim \|b^{k,m}|B_p^{a_\varepsilon,N_\varepsilon}(\mathbb{R}^n)\| \sim 2^{-\frac{\varepsilon kn}{p}}\|a^{k,m}|B_p^{T_k(a_\varepsilon),N_\varepsilon}(\mathbb{R}^n)\| \lesssim 2^{\varepsilon\alpha k},$$

where we also used (5.4.6). As $\operatorname{supp} b^{k,m} \subset B(\gamma^{k,m}, 2d2^{-\varepsilon k})$ and $a = (\alpha)\beta\boldsymbol{h}^{1/p}(n)^{1/p}$, then, up to the multiplication by convenient constants, $b^{k,m}$ are $2d$-$(\beta\boldsymbol{h}^{1/p}(n)^{1/p}, p)_\alpha$-$\varepsilon$-atoms according to Definition 2.5.1. Let

$$g = \sum_{k=0}^\infty \sum_{m=1}^{M_k} \nu_{k,m} b^{k,m}(x) \qquad \text{in } \; L_p(\mathbb{R}^n).$$

By Theorem 2.5.3,

$$g \in B_p^{\beta\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}(\mathbb{R}^n) \quad \text{and} \quad \|g|B_p^{\beta\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}(\mathbb{R}^n)\| \lesssim \|\nu|b_p^\Gamma\|.$$

Let us prove that $\operatorname{tr}_\Gamma g = f$. Let $T \in \mathbb{N}_0$. Then

$$\|f - \mathrm{tr}_\Gamma\, g|L_p(\Gamma)\| \leq \left\| f - \sum_{k=0}^{T}\sum_{m=1}^{M_k} \nu_{k,m} a_\Gamma^{k,m} |L_p(\Gamma)\right\| + \left\| \sum_{k=0}^{T}\sum_{m=1}^{M_k} \nu_{k,m} a_\Gamma^{k,m} - \mathrm{tr}_\Gamma \sum_{k=0}^{\infty}\sum_{m=1}^{M_k} \nu_{k,m} b^{k,m} |L_p(\Gamma)\right\|. \tag{3.5.5}$$

By (3.5.1), the first expression in (3.5.5) converges to 0 when $T \to \infty$. The second can be estimated as follows

$$\left\| \mathrm{tr}_\Gamma \sum_{k=T+1}^{\infty}\sum_{m=1}^{M_k} \nu_{k,m} b^{k,m} |L_p(\Gamma)\right\| \lesssim \left\| \sum_{k=T+1}^{\infty}\sum_{m=1}^{M_k} \nu_{k,m} b^{k,m} |B_p^{\beta \boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}(\mathbb{R}^n)\right\| \lesssim \left(\sum_{k=T+1}^{\infty}\sum_{m=1}^{M_k} |\nu_{k,m}|^p \right)^{\frac{1}{p}}.$$

Letting $T \to \infty$ we conclude the proof. $\square$

Remark 3.5.4. *Following the same procedure as in Remark 2.5.4 we can prove that in the conditions of Theorem 3.5.3, if* $\nu \in b_p^\Gamma$*, then*

$$\sum_{k=0}^{\infty}\sum_{m=1}^{M_k} \nu_{k,m} a_\Gamma^{k,m}(\gamma),$$

converges in $L_p(\Gamma)$*, for all families of* d*-*$(\beta_\varepsilon, p)_\Gamma^*$*-*$\varepsilon$*-atoms [respectively for* d*-*$(\beta, p, \varepsilon)_\Gamma^{**}$*-atoms]* $a_\Gamma^{k,m}$*,* $k \in \mathbb{N}_0$*,* $m \in \{1, ..., M_k\}$*.*

Chapter 4

Besov spaces of generalised smoothness on h-spaces

In this chapter Besov spaces of generalised smoothness on a class of abstract quasi-metric spaces are studied. Since some time there has been an interest in considering spaces on abstract structures that we shall describe in this chapter, giving references.
In this work we consider a class of quasi-metric spaces which we call h-spaces, imitating the notation of h-sets in $\mathbb{R}^n$. We define Besov spaces of generalised smoothness on these structures following an approach introduced by Triebel in [Tri05]: we consider convenient charts which allow us to identify these abstract structures with fractal sets in $\mathbb{R}^n$, in particular with h-sets. We define and characterise function spaces on h-spaces using charts and what is known for corresponding spaces on fractal h-sets.

This chapter is organised as follows. First we collect the notation on abstract quasi-metric spaces and we present the results which lead to the construction of the charts that we have already mentioned. Then we define Besov spaces on h-spaces, prove a characterisation with atomic decompositions and we guarantee that, in some sense, these Besov spaces are independent of the charts used in the construction. Finally we present a result on the entropy numbers of embeddings between Besov spaces on h-spaces, taking advantage of a similar result for spaces on h-sets, exemplifying how this technique allows to reduce the analysis on abstract structures to sets on $\mathbb{R}^n$.

4.1 Quasi-metric spaces and Euclidean charts

In this section we present some basic assertions about quasi-metric spaces and also the concept of *Euclidean charts*.

Definition 4.1.1. *Let X be a (non-empty) set. A function $\varrho : X \times X \to [0, \infty)$ is a quasi-metric if*

$$\varrho(x, y) = 0 \text{ if, and only if, } x = y,$$

$$\varrho(x, y) = \varrho(y, x) \text{ for all } x, y \in X,$$

there is a number $A \geq 1$ such that for all $x, y, z \in X$

$$\varrho(x, y) \leq A[\varrho(x, z) + \varrho(z, y)]. \tag{4.1.1}$$

If (4.1.1) *is true with $A = 1$ then ϱ is a metric.*

Notation 4.1.2. *In what follows we will use the following notation*

$$B^X(x, r) := \{y \in X : \varrho(x, y) < r\}, \qquad x \in X, \quad r > 0.$$

Useful properties about quasi-metrics spaces are given in the next theorem.

Theorem 4.1.3. *Let ϱ be a quasi-metric on a set X.*

(i) There is a number ε_0 with $0 < \varepsilon_0 \leq 1$ and a quasi-metric $\overline{\varrho}$ such that $\varrho \sim \overline{\varrho}$ and, for any $0 < \varepsilon \leq \varepsilon_0$, $\overline{\varrho}^\varepsilon$ is a metric.

(ii) Let $0 < \varepsilon \leq \varepsilon_0$. There is a positive number c such that for all $x \in X$, $y \in X$, $z \in X$,

$$|\overline{\varrho}(x, y) - \overline{\varrho}(x, z)| \leq c\, \overline{\varrho}(x, y)^\varepsilon \, [\overline{\varrho}(x, y) + \overline{\varrho}(x, z)]^{1-\varepsilon}. \tag{4.1.2}$$

Remark 4.1.4. *For proofs of part (i) we refer to [Hei01, p. 110-112, Proposition 14.5] and [Tri05, p. 25, Remark 3.2]. In the latter it was also remarked that, though* (4.1.2) *is known since some time (cf. [MS79, p. 259, Theorem 2]), it can also be obtained as a consequence of (i).*

Convention 4.1.5. *Let X be a non-empty set, ϱ a quasi-metric and $\overline{\varrho}$ be fixed and as in Theorem 4.1.3(ii). Hereafter we consider X equipped with the topology which is generated taking the balls*

$$B^X_{\overline{\varrho}}(x,r) = \{y \in X : \overline{\varrho}(x,y) < r\}, \quad r > 0,$$

as a basis of neighborhoods of $x \in X$.

Definition 4.1.6. *Let ϱ be a quasi-metric on a set X equipped with the topology as indicated in Convention 4.1.5.*

(i) (X,ϱ,μ) *is called a space of homogeneous type if μ is a non-negative regular Borel measure on X such that there is a constant A' with*

$$0 < \mu(B^X(x,2r)) \leq A'\,\mu(B^X(x,r)), \quad \text{for all} \quad x \in X,\ r > 0, \tag{4.1.3}$$

(doubling condition).

(ii) Let $h \in \mathbb{H}$ according to Definition 1.4.1. Then (X,ϱ,μ) is called an h-space if it is a complete space of homogeneous type as in part (i) with

$$\operatorname{Diam} X = \sup\{\varrho(x,y) : x, y \in X\} < \infty \tag{4.1.4}$$

and

$$\mu(B^X(x,r)) \sim h(r) \quad \text{for all} \quad x \in X \quad \text{and} \quad 0 < r \leq 1$$

Remark 4.1.7. *There has been since some time an interest in considering spaces of homogeneous type (cf. [CW71], [DJS85] and [Chr90]). In the last decade there have been many authors working to develop a substantial intrinsic analysis on fractals and on quasi-metric spaces. We refer to [DS97], [Hei01], [Kig01] and [Sem01].*
As far as Besov spaces on some classes of spaces of homogeneous type are concerned we refer to the survey [HY02] where these spaces were studied in detail.

Remark 4.1.8. *In the particular case where X is a subset of some $\mathbb{R}^n$ and ρ is the corresponding Euclidean metric, X is an h-set.*
Back to the general abstract case, in the particular case where $h(r) = r^d$, $r > 0$, h-spaces are called d-spaces. The notation d-spaces was introduced in [TY02] imitating the notation of d-sets in $\mathbb{R}^n$.

The next theorem paves the way to relate quasi-metric spaces and fractal sets in some $\mathbb{R}^n$.

Theorem 4.1.9. *Let (X, ϱ, μ) be a space of homogeneous type. Let $0 < \varepsilon_0 \leq 1$ be the number mentioned in Theorem 4.1.3 and let $0 < \varepsilon < \varepsilon_0$. Then there is an $n \in \mathbb{N}$ and a bi-Lipschitzian mapping*

$$L : X \to \mathbb{R}^n$$

from $(X, \varrho^\varepsilon, \mu)$ into $\mathbb{R}^n$. This means that

$$\varrho^\varepsilon(x, y) \sim |L(x) - L(y)|, \quad x, y \in X. \tag{4.1.5}$$

The dimension n and the bi-Lipschitzian constants involved in (4.1.5) can be chosen to depend only on ε and on the doubling constant A' in (4.1.3).

Remark 4.1.10. *This theorem was firstly proved by P. Assouad (cf. [Ass79, Ass83]). More information about this may be found in [Hei01].*
If ϱ is a quasi-metric and $\varepsilon < \varepsilon_0$, for ε_0 as in Theorem 4.1.3, we say that $(X, \varrho^\varepsilon, \mu)$ is a snowflaked version of (X, ϱ, μ). In particular, if ϱ is a metric, a snowflaked version of (X, ϱ, μ) is a structure $(X, \varrho^\varepsilon, \mu)$ with $\varepsilon < 1$.

Proposition 4.1.11. *Let (X, ϱ, μ) be an h-space and let $\varepsilon_0, \varepsilon$ and L be as in Theorem 4.1.9. Let $h_{1/\varepsilon}$ be given by*

$$h_{1/\varepsilon}(r) := h(r^{1/\varepsilon}), \quad r > 0. \tag{4.1.6}$$

Then $\Gamma = L(X) \subset \mathbb{R}^n$ is an $h_{1/\varepsilon}$-set.

Proof. For $r > 0$ we denote by $B_\varepsilon^X(x, r)$ the balls

$$B_\varepsilon^X(x, r) = \{y \in X : \varrho^{\,\varepsilon}(x, y) < r\}.$$

Hence, for all $x \in X$ and $r > 0$,

$$B_\varepsilon^X(x, r) = B^X(x, r^{1/\varepsilon}).$$

As L is a bi-Lipschitzian mapping from $(X, \varrho^\varepsilon, \mu)$ into $\mathbb{R}^n$, there are $0 < a_1 \leq a_2$ with

$$a_1 \varrho^{\,\varepsilon}(x, y) \leq |L(x) - L(y)| \leq a_2 \varrho^{\,\varepsilon}(x, y), \quad \text{for all } \ x, y \in X. \tag{4.1.7}$$

By (4.1.7), (4.1.4) and the assumption that X is complete it follows that $\Gamma = L(X)$ is compact.

Considering $\nu = \mu \circ L^{-1}$, by (4.1.7) we obtain

$$\nu(B(\gamma, r)) \sim h_{1/\varepsilon}(r), \quad \gamma \in \Gamma, \ 0 < r \leq 1.$$

□

Remark 4.1.12. *So, in the previous conditions, $h_{1/\varepsilon}$ is a measure function in $\mathbb{R}^n$ and $\nu \sim \mathcal{H}_\Gamma^{h_{1/\varepsilon}}$, where $\mathcal{H}_\Gamma^{h_{1/\varepsilon}}$ is the restriction of the Hausdorff measure $\mathcal{H}_\Gamma^{h_{1/\varepsilon}}$ in $\mathbb{R}^n$ to Γ.*

Definition 4.1.13. *Let (X, ϱ, μ) be an h-space and let $0 < \varepsilon < \varepsilon_0$ where ε_0 is the same number as in Theorem 4.1.3. We say that $(X, \varrho, \mu; L)$ or, for short, $(X; L)$ is an Euclidean ε-chart or an ε-chart if L is a bi-Lipschitzian map from $(X, \varrho^\varepsilon, \mu)$ onto $(\Gamma, \varrho_n, \mathcal{H}_\Gamma^{h_{1/\varepsilon}})$, where $h_{1/\varepsilon}$ is as in (4.1.6) and ϱ_n denotes the usual metric in $\mathbb{R}^n$.*

4.2 Function spaces on h-spaces

We define Besov spaces of generalised smoothness on h-spaces, using ε-charts.

Definition 4.2.1. *Let (X, ϱ, μ) be an h-space with an ε-chart $(X; L)$. Consider $\nu = \mu \circ L^{-1}$, $h_{1/\varepsilon}$ the function in (4.1.6) and $\Gamma = L(X) \subset \mathbb{R}^n$, where n is chosen large enough so that*

$$-\underline{s}(\boldsymbol{h}_{1/\varepsilon}) = -\frac{1}{\varepsilon}\,\underline{s}(\boldsymbol{h}) < n,$$

i.e., so that Γ satisfies the ball condition (we refer to Corollary 3.1.11). Let σ be an admissible sequence such that $\underline{s}(\sigma) > 0$ and consider $\sigma_{1/\varepsilon}$ according to the notation introduced in (1.2.6)*. Let $0 < p, q \leq \infty$. Then*

$$B^{\sigma}_{p,q}(X, \varrho, \mu; L) := \mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma, \varrho_n, \nu) \circ L, \tag{4.2.1}$$

i.e., $f \in B^{\sigma}_{p,q}(X, \varrho, \mu; L)$ if and only if $f = g \circ L$ for some $g \in \mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma, \varrho_n, \nu)$ and

$$\|f \,|\, B^{\sigma}_{p,q}(X, \varrho, \mu; L)\| := \|g \,|\, \mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma, \varrho_n, \nu)\|.$$

Remark 4.2.2. *We schematise the construction described:*

$$\begin{array}{ccccc} \underset{h\text{-space}}{(X, \varrho, \mu)} & \underset{\text{snowfl.}}{\rightsquigarrow} & (X, \varrho^{\varepsilon}, \mu) & \underset{L}{\longrightarrow} & \underset{h_{1/\varepsilon}\text{-set}}{(\Gamma, \varrho_n, \nu)}. \\ B^{\sigma}_{p,q}(X; L) & & & & \mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma) \end{array}$$

The space $B^{\sigma}_{p,q}(X, \varrho, \mu; L)$, or just $B^{\sigma}_{p,q}(X; L)$, is a quasi-Banach space (Banach space if $p \geq 1$ and $q \geq 1$).
If (X, ϱ, μ) is an h-set (Γ, ϱ_n, ν) in some $\mathbb{R}^n$ and if L is the identity, then it follows immediately from Definition 4.2.1 that $B^{\sigma}_{p,q}(X; L) = \mathbb{B}^{\sigma}_{p,q}(\Gamma)$. But, if we take different functions L we may introduce different scales of spaces in $(\Gamma, \varrho_n, \nu; L)$ which might possibly not be obtained from trace spaces according to Definition 3.2.4. Nevertheless, we will prove that, under some conditions with respect to h, σ and p, for a certain range of values of ε, the spaces $B^{\sigma}_{p}(X; L)$ do not depend on the ε-chart considered.

Remark 4.2.3. *As we have already mentioned, the use of Euclidean charts to define Besov spaces on quasi-metric spaces was introduced by Triebel in [Tri05]. In that paper the spaces $B^{(s)}_{p}(X, \varrho, \mu; L)$, where $s > 0$, $1 < p < \infty$ and (X, ϱ, μ) is a d-space, were defined using a kind of quarkonial decomposition (cf. [Tri05, p. 34, Definition 4.6]). The adapted quarks for $B^{(s)}_{p}(X; L)$ are the composition of the quarks for the corresponding space $\mathbb{B}^{(s/\varepsilon)}_{p}(\Gamma)$ with the ε-chart L.*
It is an immediate consequence of Definition 4.2.1 and the existence, under some restrictions, of characterisations with quarkonial decompositions for spaces $\mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma)$ (cf. [Bri01] and [KZ06]) that something analogous could be obtained for the spaces defined above.

Remark 4.2.4. *There has been an interest in considering Besov spaces of classical smoothness on d-spaces. There are several papers on these spaces, namely [HLY99], [HY02], [TY02], [Yan02] and [Yan03]. In these works Besov spaces on these abstract structures are defined intrinsically using functions which look like substitutes of local means which are used to characterise Besov spaces* $B^{(s)}_{p,q}(\mathbb{R}^n)$.
In [Yan03], Yang proved that, under some restrictions, Besov spaces defined on a d-set regarded as a d-space coincide with Besov spaces defined on d-sets using Triebel's methods, based on traces as in Definition 3.2.4.
In [Tri05], Triebel proved that, under some conditions, the Besov spaces on d-spaces by the usual approach coincide with the spaces defined by means of ε*-charts.*

In the next theorem we state that, in some sense and under some restrictions, the Besov spaces on h-spaces are independent of the charts considered.

Theorem 4.2.5. *Let* $1 < p \leq \infty$ *and* σ *be an admissible sequence. For* $i \in \{1, 2\}$, *let* $(X, \varrho, \mu; L_i)$ *be* ε_i*-charts of an h-space* (X, ϱ, μ). *If*

$$\overline{s}(\boldsymbol{h}) < 0 \quad \textit{and} \quad 0 < \underline{s}(\sigma) \leq \overline{s}(\sigma) < \min\{\varepsilon_1, \varepsilon_2\}, \tag{4.2.2}$$

then

$$B^\sigma_p(X; L_1) = B^\sigma_p(X; L_2). \tag{4.2.3}$$

Proof. We recall that, for $i = 1, 2$,

$$\begin{array}{ccccc} \underset{h\text{-space}}{(X, \varrho, \mu)} & \underset{\text{snowfl.}}{\rightsquigarrow} & (X, \varrho^{\varepsilon_i}, \mu) & \underset{L_i}{\longrightarrow} & \underset{h_{1/\varepsilon_i}\text{-set}}{(\Gamma_i, \varrho_{n_i}, \nu_i)} \quad , \\ B^\sigma_p(X; L_i) & & & & \mathbb{B}^{\sigma_{1/\varepsilon_i}}_p(\Gamma_i, \varrho_{n_i}, \nu_i) \end{array}$$

where

$$\Gamma_i = L_i(X) \subset \mathbb{R}^{n_i}, \qquad \nu_i = \mu \circ L_i^{-1}$$

and ϱ_{n_i} denotes the usual metric in $\mathbb{R}^{n_i}$. As previously, n_i, $i = 1, 2$, are chosen conveniently large such that

$$-\frac{1}{\varepsilon_i}\,\underline{s}(\boldsymbol{h}) < n_i.$$

By (4.2.1), (4.2.3) is equivalent to

$$\mathbb{B}_p^{\sigma_{1/\varepsilon_1}}(\Gamma_1, \varrho_{n_1}, \nu_1) = \mathbb{B}_p^{\sigma_{1/\varepsilon_2}}(\Gamma_2, \varrho_{n_2}, \nu_2) \circ L_2 \circ L_1^{-1}. \tag{4.2.4}$$

Let us prove that, given $g_1 \in \mathbb{B}_p^{\sigma_{1/\varepsilon_1}}(\Gamma_1, \varrho_{n_1}, \nu_1)$, there is $g_2 \in \mathbb{B}_p^{\sigma_{1/\varepsilon_2}}(\Gamma_2, \varrho_{n_2}, \nu_2)$ such that

$$g_1 = g_2 \circ L_2 \circ L_1^{-1} \quad \text{in} \quad \Gamma_1$$

and

$$\|g_2|\mathbb{B}_p^{\sigma_{1/\varepsilon_2}}(\Gamma_2, \varrho_{n_2}, \nu_2)\| \lesssim \|g_1|\mathbb{B}_p^{\sigma_{1/\varepsilon_1}}(\Gamma_1, \varrho_{n_1}, \nu_1)\|. \tag{4.2.5}$$

For all $\gamma_1, \delta_1 \in \Gamma_1$,

$$\varrho_{n_2}(L_2 \circ L_1^{-1}(\gamma_1), L_2 \circ L_1^{-1}(\delta_1)) \sim \varrho_{n_1}^{\varepsilon_2/\varepsilon_1}(\gamma_1, \delta_1). \tag{4.2.6}$$

We assume that $\varepsilon_2 \le \varepsilon_1$ and fix, for all $j \in \mathbb{N}_0$,

$$\{\gamma^{j,m} : m = 1, ..., M_j\} \subset \Gamma_2$$

$2^{-\varepsilon_2/\varepsilon_1}$-approximate lattices for Γ_2. It follows from Remark 3.3.2 that

$$M_j \sim h_{1/\varepsilon_2}(2^{-\frac{\varepsilon_2}{\varepsilon_1}j}) = h(2^{-\frac{j}{\varepsilon_1}}), \quad j \in \mathbb{N}_0.$$

By (4.2.6), for all $j \in \mathbb{N}_0$,

$$\{\delta^{j,m} : m = 1, ..., M_j\} \quad \text{with} \quad \delta^{j,m} = (L_1 \circ L_2^{-1})(\gamma^{j,m}),$$

are 2^{-j}-approximate lattices for Γ_1.

Let $g_1 \in \mathbb{B}_p^{\sigma_{1/\varepsilon_1}}(\Gamma_1, \varrho_{n_1}, \nu_1)$. By Proposition 1.2.15 and (4.2.2)

$$0 < \underline{s}(\sigma_{1/\varepsilon_1}) \le \overline{s}(\sigma_{1/\varepsilon_1}) < \frac{\varepsilon_2}{\varepsilon_1}.$$

Hence, applying Theorem 3.5.3, there is $\lambda \in b_p^{\Gamma_1}$ such that

$$g_1 = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_{j,m} a_{\Gamma_1}^{j,m} \quad \text{in } L_p(\Gamma_1, \nu_1),$$

where, for $j \in \mathbb{N}_0$ and $m \in \{1, ..., M_j\}$, $a^{j,m}_{\Gamma_1}$ are $\varepsilon_2/\varepsilon_1$-$d$-$(\sigma_{1/\varepsilon_1}, p)^{**}_{\Gamma_1}$-atoms located in $B(\delta^{j,m}, d2^{-j})$, for a conveniently chosen d, and

$$\|\lambda | b^{\Gamma_1}_p\| \lesssim \|g_1 | \mathbb{B}^{\sigma_{1/\varepsilon_1}}_p(\Gamma_1, \varrho_{n_1}, \nu_1)\|.$$

Let

$$a^{j,m}_{\Gamma_2} := a^{j,m}_{\Gamma_1} \circ L_1 \circ L_2^{-1}, \quad j \in \mathbb{N}_0, \quad m \in \{1, ..., M_j\}.$$

The functions $a^{j,m}_{\Gamma_2}$ are $\varepsilon_2/\varepsilon_1$-$d'$-$(\sigma_{1/\varepsilon_2}, p)^{*}_{\Gamma_2}$-atoms located at $B(\gamma^{j,m}, d'2^{-\varepsilon_2/\varepsilon_1 j})$, for some conveniently chosen d'. Then, according to Remark 3.5.4,

$$\sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_{j,m} a^{j,m}_{\Gamma_2}$$

converges in $L_p(\Gamma_2, \nu_2)$ to, say, g_2. Again by Theorem 3.5.3, we conclude that $g_2 \in \mathbb{B}^{\sigma_{1/\varepsilon_2}}_p(\Gamma_2, \varrho_{n_2}, \nu_2)$ and

$$\|g_2 | \mathbb{B}^{\sigma_{1/\varepsilon_2}}_p(\Gamma_2, \varrho_{n_2}, \nu_2)\| \lesssim \|\lambda | b^{\Gamma_1}_p\|.$$

Hence $g_2 = g_1 \circ L_1 \circ L_2^{-1}$ and (4.2.5) follows.
The reverse inclusion in (4.2.4) is proved analogously. □

Remark 4.2.6. *We can interprete $L_2 \circ L_1^{-1}$ as being an $\varepsilon_2/\varepsilon_1$-chart of the h_{1/ε_1}-space $(\Gamma_1, \varrho_{n_1}, \nu_1)$. Hence,* (4.2.4) *can be written as*

$$B^{\sigma_{1/\varepsilon_1}}_p(\Gamma_1; L_2 \circ L_1^{-1}) = \mathbb{B}^{\sigma_{1/\varepsilon_1}}_p(\Gamma_1),$$

based on

$$\begin{array}{ccccc} \underset{h_{1/\varepsilon_1}\text{-space}}{(\Gamma_1, \varrho_{n_1}, \nu_1)} & \underset{\text{snowfl.}}{\rightsquigarrow} & (\Gamma_1, \varrho^{\varepsilon_2/\varepsilon_1}_{n_1}, \nu_1) & \underset{L_2 \circ L_1^{-1}}{\longrightarrow} & \underset{h_{1/\varepsilon_2}\text{-set}}{(\Gamma_2, \varrho_{n_2}, \nu_2)}. \\ B^{\sigma_{1/\varepsilon_1}}_p(\Gamma_1; L_2 \circ L_1^{-1}) & & & & \mathbb{B}^{\sigma_{1/\varepsilon_2}}_p(\Gamma_2) \end{array}$$

So, to prove that the definition of Besov spaces on abstract h-spaces using ε-charts is independent of the charts, under some restrictions, it was enough to prove that this construction works (in the sense of being independent of the charts) in the particular case of h-sets.

The next definition corresponds to the abstract version of Definition 3.3.1.

Definition 4.2.7. *Let (X,ϱ,μ) be an h-space and $j\in\mathbb{N}_0$. We say that*

$$\{x^{j,m}\}_{m=1}^{M_j}\subset X$$

with $j\in\mathbb{N}_0$, is a $2^{-j}-$approximate lattices for X if there exist positive numbers c_1 and c_2 independent of j such that

$$\varrho(x^{j,m_1},x^{j,m_2})\geq c_1 2^{-j},\quad j\in\mathbb{N}_0,\quad m_1\neq m_2,$$

$$X=\bigcup_{m=1}^{M_j}B_{j,m}\quad \textit{with}\quad B_{j,m}=B^X(x^{j,m},c_2 2^{-j})\quad \textit{for } j\in\mathbb{N}_0. \tag{4.2.7}$$

Remark 4.2.8. *Let us note that if we consider an Euclidean ε-chart of (X,ϱ,μ), L, with $L(X)=\Gamma$, then*

$$\gamma_1,\gamma_2\in\Gamma\quad\Leftrightarrow\quad\gamma_1=L(x_1)\quad \textit{and}\quad \gamma_2=L(x_2),\quad x_1,x_2\in X$$

and so

$$|\gamma_1-\gamma_2|\sim 2^{-\varepsilon j}\quad\Leftrightarrow\quad \varrho(x_1,x_2)\sim 2^{-j}$$

As a result, the existence of 2^{-j}-approximate lattices for X follows from the existence of ε-charts and $2^{-\varepsilon j}$-approximate lattices for corresponding sets in $\mathbb{R}^n$ as in Definition 3.3.1. If, for $j\in\mathbb{N}_0$,

$$\{\gamma^{j,m}:m=1,...,M_j\}\subset\Gamma$$

is a $2^{-\varepsilon j}$-approximate lattice for the $h_{1/\varepsilon}$-set Γ, then

$$\{x^{j,m}:m=1,...,M_j\}\subset X\qquad \textit{with}\quad x^{j,m}=L^{-1}(\gamma^{j,m})$$

is a 2^{-j}-approximate lattice for X. Moreover, according to Remark 3.3.2, for all $j\in\mathbb{N}_0$,

$$M_j\sim h_{1/\varepsilon}(2^{-\varepsilon j})^{-1}=h(2^{-j})^{-1}.$$

We also define an abstract version of atoms.

Definition 4.2.9. *Let $h\in\mathbb{H}$, (X,ϱ,μ) be an h-space and $\varepsilon\in(0,\varepsilon_0)$, for ε_0 as in Theorem 4.1.3. Let $\{x_{j,m}\}_{m=1}^{M_j}$, $j\in\mathbb{N}_0$, be 2^{-j}-approximate lattices for X and $d>c_2$, where c_2 is as* *in* (4.2.7). *Consider an admissible sequence* σ *and* $1 < p \leq \infty$. *A function on* X, $a_X^{j,m}$, *is called an* d-$(\sigma, p, \varepsilon)_X$*-atom if for* $j \in \mathbb{N}_0$ *and* $m = 1, ..., M_j$,

(a) $\operatorname{supp} a_X^{j,m} \subset B(x^{j,m}, d2^{-j})$

(b) $|a_X^{j,m}(x)| \leq \sigma_j^{-1}\, h(2^{-j})^{-\frac{1}{p}}, \qquad x \in X,$

(c) $|a_X^{j,m}(x) - a_X^{j,m}(y)| \leq \sigma_j^{-1}\, h(2^{-j})^{-\frac{1}{p}} 2^{\varepsilon j} \varrho^{\varepsilon}(x, y), \qquad x, y \in X.$

The next theorem states a characterisation of Besov spaces on h-spaces with atomic decompositions.

Theorem 4.2.10. *Let* $(X, \varrho, \mu; L)$ *be an* ε*-chart of an* h*-space* (X, ϱ, μ). *Let* $1 < p \leq \infty$ *and* σ *be an admissible sequence. Assume that*

$$\overline{s}(\boldsymbol{h}) < 0 \quad \textit{and} \quad 0 < \underline{s}(\sigma) \leq \overline{s}(\sigma) < \varepsilon. \tag{4.2.8}$$

Let $d > c_2$, *where* c_2 *is as in* (4.2.7). *Then* $B_p^\sigma(X; L)$ *is the collection of all* $f \in L_p(X)$ *which can be represented as*

$$f = \sum_{j=1}^{\infty} \sum_{m=1}^{M_j} \lambda_{j,m} a_X^{j,m}(x), \qquad \textit{in} \quad L_p(X), \tag{4.2.9}$$

for some $\lambda \in b_p^{L(X)}$, *where* $a_X^{j,m}$ *are* d-$(\sigma, p, \varepsilon)_X$*-atoms according to Definition 4.2.9. Furthermore,*

$$\|f|B_p^\sigma(X; L)\| \sim \inf \|\lambda|b_p^{L(X)}\|,$$

where the infimum is taken over all representations (4.2.9).

Proof. One can easily see that $a_X^{j,m}$ is a d-$(\sigma, p, \varepsilon)_X$-atom if, and only if, $a_X^{j,m} \circ L^{-1}$ is a d'-$(\sigma_{1/\varepsilon}, p)_\Gamma^*$-$\varepsilon$-atom, where $\Gamma = L(X)$ is an $h_{1/\varepsilon}$-set and d' is conveniently chosen. Hence, the above result can be obtained using ε-charts and applying Theorem 3.5.3 to the spaces $\mathbb{B}_p^{\sigma_{1/\varepsilon}}(\Gamma)$. □

Remark 4.2.11. *If $h(r) = r^d$ and $\sigma = (s)$, then* (4.2.8) *corresponds to assuming $d > 0$ and*

$$0 < s < \varepsilon,$$

which coincides with the conditions obtained by Triebel in [Tri05, p. 42, Theorem 4.22]. In this work, to guarantee the uniqueness of the spaces $B_p^{(s)}(X; L)$ (where X is a d-space), instead of a direct proof as it was done in Theorem 4.2.5, transferring everything to function spaces on special sets in $\mathbb{R}^n$, Triebel used a result corresponding to the above one to conclude that the spaces $B_p^{(s)}(X; L)$ coincide with the spaces considered by Han and Yang in [HY02] and, consequently, are independent of the Euclidean charts.

4.3 Example: entropy numbers

In this section we will present an example which shows that this approach for the definition of the function spaces allows to develop a theory for function spaces on quasi-metric spaces, taking advantage of what is already known for function spaces on fractals in $\mathbb{R}^n$. First we recall the notion of entropy number of an operator.

Definition 4.3.1. *Let A and B be two quasi-Banach spaces and let $T : A \to B$ be a linear and bounded operator. Then for all $j \in \mathbb{N}$, the j-th (dyadic) entropy number of T is defined by*

$$e_j(T) = \inf\{\delta > 0 : T(\mathcal{B}_A) \subset \bigcup_{l=1}^{2^{j-1}} (b_l + \delta\mathcal{B}_{\mathcal{B}}), \textit{ for some } b_1, ..., b_{2^{j-1}} \in B\},$$

where $\mathcal{B}_A$ and $\mathcal{B}_B$ denote the closed unitary balls in A and in B, respectively.

Remark 4.3.2. *If $(\alpha_j)_{j\in\mathbb{N}}$ is an increasing sequence of positive numbers we write e_{α_j} instead of $e_{[\alpha_j]}$, where $[\cdot]$ denotes the integer-part function.*

In the next Proposition we present estimates for the entropy numbers of embeddings between function spaces on h-spaces.

Proposition 4.3.3. *Let $h \in \mathbb{H}$ be such that $\overline{s}(\boldsymbol{h}) < 0$. Consider an h-space (X, ϱ, μ) with an ε-chart $(X; L)$, according to Definition 4.1.13.*

Let σ and τ be admissible sequences such that $0 < \underline{s}(\tau) < \underline{s}(\sigma)$. Consider $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ and

$$\underline{s}(\sigma\tau^{-1}) > -\underline{s}(\boldsymbol{h})\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+.$$

Then the embedding

$$id^X : B^{\sigma}_{p_1,q_1}(X; L) \hookrightarrow B^{\tau}_{p_2,q_2}(X; L)$$

is compact and

$$e_{h_j^{-1}}(id^X) \sim \tau_j \sigma_j^{-1}. \tag{4.3.1}$$

Proof. Using ε-charts this result is just a consequence of a corresponding one for Besov spaces on h-sets, which was proved by Bricchi (cf. [Bri01, p. 130, Theorem 4.3.2]). In the application of this theorem it may be convenient to choose n sufficiently large, where n stands for the dimension of the Euclidean space which contains $L(X)$.

By (4.2.1)

$$\mathscr{L} : f \mapsto f \circ L \ : \ \mathbb{B}^{\sigma_{1/\varepsilon}}_{p,q}(\Gamma, \varrho_n, \nu) \ \hookrightarrow \ B^{\sigma}_{p,q}(X, \varrho, \mu; L)$$

is an isomorphic map.

We decompose id^X according to the following commutative diagram.

$$\begin{array}{ccc} B^{\sigma}_{p_1,q_1}(X; L) & \xrightarrow{\mathscr{L}^{-1}} & \mathbb{B}^{\sigma_{1/\varepsilon}}_{p_1,q_1}(\Gamma, \varrho_n, \nu) \\ id^X \Big\downarrow & & \Big\downarrow id^\Gamma \\ B^{\tau}_{p_2,q_2}(X; L) & \xleftarrow[\mathscr{L}]{} & \mathbb{B}^{\tau_{1/\varepsilon}}_{p_2,q_2}(\Gamma, \varrho_n, \nu) \end{array}$$

Hence,

$$e_{h_j^{-1}}(id^X) \sim e_{h_j^{-1}}(id^\Gamma).$$

By [Bri01, p. 130, Theorem 4.3.2],

$$e_{h^{-1}_{1/\varepsilon,j}}(id^\Gamma) \sim \tau_{1/\varepsilon,j}\ \sigma^{-1}_{1/\varepsilon,j},$$

and, after some calculations, we obtain (4.3.1). □

Chapter 5

Spectral theory for the fractal Laplacian

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ such that $\Gamma \subset \Omega$ and $-\Delta$ be the Dirichlet Laplacian in Ω. In this chapter we study the operator

$$B := (-\Delta)^{-1} \circ \mathrm{tr}^{\Gamma}, \tag{5.0.1}$$

acting in convenient function spaces in Ω, where $(-\Delta)^{-1}$ is the inverse of the Dirichlet Laplacian in Ω and

$$\mathrm{tr}^{\Gamma} = \mathrm{id}^{\Gamma} \circ \mathrm{tr}_{\Gamma}, \tag{5.0.2}$$

for tr_{Γ} as in Definition 3.2.1. The operator id^{Γ}, which will be formally defined later, identifies elements of $L_p(\Gamma)$ with tempered distributions.

The operator B was studied by Triebel in the case where Γ is d-set. We refer to [Tri97, Chapter 5] and [Tri01, Chapter 3]. The case where Γ is a (d, ψ)-set was studied by Edmunds and Triebel in [ET98, ET99] and by Moura in [Mou01b]. As it was described in these works, in the case $n = 2$ the operator B has physical interest, describing the vibration of a drum where the whole mass is distributed on Γ. This is why the study of this subject is usually called the *fractal drum* problem. So we study the *fractal drum* problem in the context of h-sets, extending the results for d-sets and (d, ψ)-sets.

Making use of the results for the operator B we also prove the existence of a solution for the fractal Dirichlet problem for h-sets, i.e., we prove that, under some conditions, given g in a convenient function space on an h-set $\Gamma \subset \Omega$, there exists u such that

$$u \in H^1(\Omega), \quad \Delta u(x) = 0 \quad \text{in} \quad \Omega \setminus \Gamma,$$

and

$$\operatorname{tr}_{\partial\Omega} u = 0, \quad \operatorname{tr}_\Gamma u = g.$$

This problem was studied by Triebel for the case of d-sets (cf. [Tri01, Chapter 3, Section 20]).

In the work developed to study the operator B some other interesting results for Besov spaces of generalised smoothness were obtained, namely: a result on pointwise multipliers for these spaces and the existence of a universal extension operator acting from Besov spaces on a class of domains into corresponding function spaces on $\mathbb{R}^n$. In the proof of some of these results we used interpolation, an important tool in the theory of function spaces.

So we start this chapter collecting some notation on interpolation. Then we present results on pointwise multipliers for general Besov spaces both on $\mathbb{R}^n$ and on domains. In Section 5.3 we prove the existence of a universal extension operator and in the next Section we apply it to study the Laplacian in the context of generalised smoothness. After collecting some more results on h-sets we study the operator B, first acting in Besov spaces of generalised smoothness on Ω and then acting in $\mathring{H}^1(\Omega)$, where Ω denotes a bounded smooth domain. Finally, we consider a fractal Dirichlet problem in the context of h-sets.

The next definition complements Definition 2.4.4, where some Besov spaces on domains were introduced.

Definition 5.0.4. *Let Ω be a domain in $\mathbb{R}^n$.*

(i) Let σ be an admissible sequence, $A \in \{B, F\}$ and $0 < p \leq \infty$, $0 < q \leq \infty$, with $p < \infty$ if $A = F$. Then $A^\sigma_{p,q}(\Omega)$ is the collection of all $f \in D'(\Omega)$ such that there is a

$g \in A^{\sigma}_{p,q}(\mathbb{R}^n)$ with $g|_{\Omega} = f$. Furthermore,

$$\|f|A^{\sigma}_{p,q}(\Omega)\| := \inf \|g|A^{\sigma}_{p,q}(\mathbb{R}^n)\|, \tag{5.0.3}$$

where the infimum is taken over all $g \in A^{\sigma}_{p,q}(\mathbb{R}^n)$ such that its restriction $g|_{\Omega}$ to Ω coincides in $D'(\Omega)$ with f.

(ii) Let σ be an admissible sequence, $A \in \{B, F\}$ and $0 < p \leq \infty$, $0 < q \leq \infty$, with $p < \infty$ if $A = F$. Then $\mathring{A}^{\sigma}_{p,q}(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ in $A^{\sigma}_{p,q}(\Omega)$

Remark 5.0.5. *If Ω is a domain in $\mathbb{R}^n$, $1 < p < \infty$, $s \in \mathbb{R}$ and $m \in N$, then the meaning of*

$$W^m_p(\Omega), \quad \mathring{W}^m_p(\Omega), \quad H^s_p(\Omega) \quad \text{and} \quad \mathring{H}^s_p(\Omega)$$

follows immediately from Remark 2.1.3 and Definition 5.0.4.
If $p = 2$ we abbreviate the notation in the usual way:

$$H^s(\Omega) := H^s_2(\Omega) \quad \text{and} \quad \mathring{H}^s(\Omega) := \mathring{H}^s_2(\Omega).$$

Remark 5.0.6. *One can easily verify that* (2.1.2) *and* (2.1.3) *in Proposition 2.1.5 remain valid if we write Ω instead of $\mathbb{R}^n$.*

5.1 Interpolation with function parameter

In this section we collect some notation and results on interpolation with function parameter. In [Mer84] and [CF88] it was proved that Besov spaces of generalised smoothness can be obtained by interpolation of classic Besov spaces. In this chapter we apply this to obtain some properties for general Besov spaces, combining the known results for classic Besov spaces with interpolation. This technique was already used by other authors to extend results from classical to generalised smoothness. We refer to [CM04] and [Alm05b].

First we introduce some basic notation related to interpolation. For a detailed description we refer to [Tri95].

We consider $\{A_0, A_1\}$ an interpolation couple, meaning that A_0 and A_1 are quasi-normed spaces both continuously embedded in some Hausdorff topological vector space. In this case

$$\|a|A_0 \cap A_1\| := \max\{\|a|A_0\|, \|a|A_1\|\}, \quad a \in A_0 \cap A_1,$$

and

$$\|a|A_0 + A_1\| := \inf_{\substack{a=a_0+a_1 \\ a_0\in A_0,\ a_1\in A_1}} (\|a_0|A_0\| + \|a_1|A_1\|), \quad a \in A_0 + A_1,$$

define quasi-norms in $A_0 \cap A_1$ and $A_0 + A_1$, respectively. There are different approaches to construct interpolation spaces. In this work we use the real method based on the K-functional introduced by Peetre. This functional is defined by

$$K(t, a) = K(t, a, A_0, A_1) := \inf_{\substack{a=a_0+a_1 \\ a_0\in A_0,\ a_1\in A_1}} (\|a_0|A_0\| + t\|a_1|A_1\|). \tag{5.1.1}$$

For all $t > 0$, $K(t, \cdot, A_0, A_1)$ is an equivalent quasi-norm in the space $A_0 + A_1$.

In the next definition we consider a function in $\mathcal{B}$. This class of functions was presented in Definition 1.2.8(ii).

Definition 5.1.1. *Let $\{A_0, A_1\}$ be an interpolation couple. Let $\gamma \in \mathcal{B}$ and $0 < q \le \infty$. We define the corresponding interpolation space with function parameter by*

$$(A_0, A_1)_{\gamma,q} := \Big\{a \in A_0{+}A_1 \ : \ \|a|(A_0, A_1)_{\gamma,q}\| = \Big(\int_0^\infty [\gamma(t)^{-1}K(t, a, A_0, A_1)]^q \frac{dt}{t}\Big)^{1/q} < \infty\Big\},$$

with the usual modification if q is not finite.

By [Mer84, p. 194, Theorem 13], complemented by [CF88, p.166, Theorem 5.3, Remark 5.4] and [Alm05a, p. 30, Theorem 2.3.7], Besov spaces of generalised smoothness can be obtained by interpolation of classic Besov spaces with a convenient function parameter. This result is formally stated in the next theorem.

Theorem 5.1.2. *Let σ be an admissible sequence and $\phi \in \mathbf{B}_\sigma$. Let $0 < p \le \infty$ and $0 < q_0, q_1, q \le \infty$. Let $s_0, s_1 \in \mathbb{R}$ satisfy $s_1 < \underline{s}(\sigma) \le \overline{s}(\sigma) < s_0$ and γ be given by*

$$\gamma(t) = \frac{t^{\frac{s_0}{s_0-s_1}}}{\phi\big(t^{\frac{1}{s_0-s_1}}\big)}, \quad t \in (0, \infty). \tag{5.1.2}$$

Then

$$\left(B^{(s_0)}_{p,q_0}(\mathbb{R}^n), B^{(s_1)}_{p,q_1}(\mathbb{R}^n)\right)_{\gamma,q} = B^{\sigma}_{p,q}(\mathbb{R}^n). \tag{5.1.3}$$

Remark 5.1.3. *Usually the results related to interpolation with a function parameter are stated in terms of functions in class $\mathcal{B}$, i.e., given $\phi \in \mathcal{B}$,*

$$\left(B^{(s_0)}_{p,q_0}(\mathbb{R}^n), B^{(s_1)}_{p,q_1}(\mathbb{R}^n)\right)_{\gamma,q} = B^{(\phi(2^j))_j}_{p,q}(\mathbb{R}^n), \quad s_1 < \underline{S}(\phi) \leq \overline{S}(\phi) < s_0, \tag{5.1.4}$$

for p, p_i, q, q_i, $i = 0, 1$ as in Theorem 5.1.2 and γ as in (5.1.2). *We state the results in terms of admissible sequences, σ. This formultation follows immediately from the classic one in* (5.1.4), *if one considers a function in $\mathbf{B}_\sigma$ (we refer to Definition 1.2.12) and applies Proposition 1.2.20.*

5.2 Pointwise multipliers for Besov spaces of generalised smoothness

This section is devoted to the study of multiplication properties of Besov spaces of generalised smoothness on $\mathbb{R}^n$, first, and then on domains. If g is a function on $\mathbb{R}^n$ (respectively Ω) we study wether $f \mapsto gf$ (pointwise multiplication) yields a bounded linear mapping from Besov spaces of generalised smoothness on $\mathbb{R}^n$ (respectively Ω) into themselves. For the study of these properties in the case of Besov spaces with classical smoothness we refer to [Tri83], in particular to Section 2.8 for spaces on $\mathbb{R}^n$ and to Section 3.3.2 in the case of spaces on domains. In Section 2.8 Triebel describes in particular what is meant by gf (pointwise multiplication) when both g and f denote tempered distributions. It is also remarked that for $g \in \mathcal{S}(\mathbb{R}^n)$ the meaning of gf according to this interpretation coincides with the usual one. In the present work we shall only consider $g \in \mathcal{S}(\mathbb{R}^n)$ and so gf has the usual meaning. We study pointwise multipliers for Besov spaces of generalised smoothness applying an important tool in the theory of function spaces: interpolation.

The next theorem is based on [Tri83, p. 140, Theorem 2.8.2(i)]. In [Tri83] the pointwise multipliers were considered in spaces $B^{(\rho)}_{\infty,\infty}$, for ρ satisfying convenient conditions. In this work we consider a smaller class, $\mathcal{S}(\mathbb{R}^n)$, but we recall that this space is continuously embedded in the spaces $B^{(s)}_{p,q}(\mathbb{R}^n)$, for all $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.

Theorem 5.2.1. *Let σ be an admissible sequence, $0 < p, q \leq \infty$ and ρ satisfy*

$$\rho > \max\left\{\overline{s}(\sigma), \frac{n}{p} - \underline{s}(\sigma)\right\}. \tag{5.2.1}$$

Then any $g \in \mathcal{S}(\mathbb{R}^n)$ is a multiplier for $B^{\sigma}_{p,q}(\mathbb{R}^n)$, i.e., $f \mapsto gf$ yields a bounded linear mapping from $B^{\sigma}_{p,q}(\mathbb{R}^n)$ into itself and there is a positive constant c such that

$$\|gf|B^{\sigma}_{p,q}(\mathbb{R}^n)\| \leq c\|g|B^{(\rho)}_{\infty,\infty}(\mathbb{R}^n)\| \cdot \|f|B^{\sigma}_{p,q}(\mathbb{R}^n)\|, \tag{5.2.2}$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$ and all $f \in B^{\sigma}_{p,q}(\mathbb{R}^n)$.

Proof. We fix ρ as in (5.2.1) and we consider s_0 and s_1 such that

$$\overline{s}(\sigma) < s_0 < \rho \quad \text{and} \quad \frac{n}{p} - \rho < s_1 < \underline{s}(\sigma). \tag{5.2.3}$$

Condition (5.2.3) guarantees that the pointwise multipler theorem [Tri83, p. 140, Theorem 2.8.2(i)] can be applied to spaces $B^{(s_i)}_{p,q}(\mathbb{R}^n)$, $i = 0, 1$, and also that, for γ as in (5.1.2), (5.1.3) holds. Let us temporarily denote the spaces $B^{(s_i)}_{p,q}(\mathbb{R}^n)$ by A_i, $i = 0, 1$. Let $g \in \mathcal{S}(\mathbb{R}^n)$ and $f \in B^{\sigma}_{p,q}(\mathbb{R}^n)$. Then there are $f_i \in A_i$, $i = 0, 1$, such that $f = f_0 + f_1$. By [Tri83, p. 140, Theorem 2.8.2(i)] $f_i g \in A_i$, $i = 0, 1$, and there is a positive number c, independent of g and f_i, $i = 0, 1$, for which

$$\|gf_i|A_i\| \leq c\|g|B^{(\rho)}_{\infty,\infty}(\mathbb{R}^n)\| \cdot \|f_i|A_i\|, \quad i = 0, 1.$$

Therefore $gf \in A_0 + A_1$. For all $t > 0$, applying again the pointwise multiplier Theorem for classic Besov spaces,

$$\begin{aligned}
K(t, gf, A_0, A_1) &= \inf_{\substack{gf = a_0 + a_1 \\ a_0 \in A_0,\, a_1 \in A_1}} (\|a_0|A_0\| + t\|a_1|A_1\|) \\
&\leq \inf_{\substack{f = a_0 + a_1 \\ a_0 \in A_0,\, a_1 \in A_1}} (\|ga_0|A_0\| + t\|ga_1|A_1\|) \\
&\leq c\|g|B^{(\rho)}_{\infty,\infty}(\mathbb{R}^n)\| \inf_{\substack{f = a_0 + a_1 \\ a_0 \in A_0,\, a_1 \in A_1}} (\|a_0|A_0\| + t\|a_1|A_1\|) \\
&= c\|g|B^{(\rho)}_{\infty,\infty}(\mathbb{R}^n)\| K(t, f, A_0, A_1)
\end{aligned}$$

Hence

$$\|gf|(A_0,A_1)_{\gamma,q}\| \leq c\|g|B^{(\rho)}_{\infty,\infty}(\mathbb{R}^n)\|\|f|(A_0,A_1)_{\gamma,q}\| \sim c\|g|B^{(\rho)}_{\infty,\infty}(\mathbb{R}^n)\|\|f|B^{\sigma}_{p,q}(\mathbb{R}^n)\| < \infty.$$

So by Theorem 5.1.2, $gf \in B^{\sigma}_{p,q}(\mathbb{R}^n)$ and (5.2.2) holds, concluding the proof. $\square$

A corresponding result for Besov spaces of generalised smoothness on domains follows immediately from Definition 5.0.4(i) and Theorem 5.2.1.

Corollary 5.2.2. *Let Ω be a domain in $\mathbb{R}^n$. Let σ be an admissible sequence, $0 < p, q \leq \infty$ and ρ satisfy (5.2.1). Then any $g \in \mathcal{D}(\Omega)$ is a multiplier for $B^{\sigma}_{p,q}(\Omega)$, i.e., $f \mapsto gf$ yields a bounded linear mapping from $B^{\sigma}_{p,q}(\Omega)$ into itself and there is a positive constant c such that*

$$\|gf|B^{\sigma}_{p,q}(\Omega)\| \leq c\|g_\Omega\|B^{(\rho)}_{\infty,\infty}(\mathbb{R}^n)| \cdot \|f|B^{\sigma}_{p,q}(\Omega)\|, \tag{5.2.4}$$

for all $g \in \mathcal{D}(\Omega)$ and all $f \in B^{\sigma}_{p,q}(\mathbb{R}^n)$, where we are writing g_Ω to denote the extension by zero of the function g.

5.3 A universal extension operator

In this section we prove the existence of a universal extension operator acting from Besov spaces of generalised smoothness on Lipschitz domains to corresponding spaces on $\mathbb{R}^n$. First we present part of the notation used in the proof and some auxiliary results.

Definition 5.3.1. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$ ad $f \in \mathcal{S}(\mathbb{R}^n)$. We define $f * T$ by*

$$< f * T, \varphi > := < T, f(-\cdot) * \varphi >, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Remark 5.3.2. *For this definition we refer, for example, to [Kho72, Tome II, p. 18], where it is mentioned that $f * T$ belongs to $\mathcal{S}'(\mathbb{R}^n)$. Applying a collection of results which can be found in [Kho72], namely the fact that $D(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$, a Fubini-type property for tensor product of distributions and a characterisation for the convolution of* *distributions with functions in* $D(\mathbb{R}^n)$*, one can prove that* $f * T$ *is a function, in the sense that it corresponds to a regular distribution given by the function defined by*

$$\psi(x) :=< T, f(x - \cdot) >, \quad x \in \mathbb{R}^n.$$

Notation 5.3.3. *In this section, for* $f \in \mathcal{S}(\mathbb{R}^n)$ *and* $0 < t \leq 1$*, if nothing additionally is said,* f_t *denotes the function defined by*

$$f_t(x) = t^{-n} f(t^{-1}x), \quad x \in \mathbb{R}^n.$$

Definition 5.3.4. *Let* $K \geq -1$ *be an integer,* $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ *and* $r > 0$*. Let* $g \in \mathcal{S}'(\mathbb{R}^n)$*. We define*

$$(\varphi_0^* g)_r(x) := \sup_{y \in \mathbb{R}^n} \frac{|\varphi_0 * g(y)|}{(1 + |x - y|)^r}, \quad x \in \mathbb{R}^n,$$

and, for $j \in \mathbb{N}$*,*

$$(\varphi_{2^{-j}}^* g)_r(x) := \sup_{y \in \mathbb{R}^n} \frac{|\varphi_{2^{-j}} * g(y)|}{(1 + 2^j|x - y|)^r}, \quad x \in \mathbb{R}^n,$$

where $\varphi_{2^{-j}}$ *are according to Notation 5.3.3. Additionally, consider an admissible sequence* σ *and* $0 < p, q \leq \infty$*. We define*

$$\|g|B_{p,q}^{\sigma}(\mathbb{R}^n)\|_{\varphi_0,\varphi,r}^{(1)} := \sigma_0 \|(\varphi_0^* g)_r|L_p(\mathbb{R}^n)\| + \Big(\sum_{j=1}^{\infty} \sigma_j^q \|(\varphi_{2^{-j}}^* g)_r|L_p(\mathbb{R}^n)\|^q \Big)^{1/q}$$

and

$$\|g|B_{p,q}^{\sigma}(\mathbb{R}^n)\|_{\varphi_0,\varphi,r}^{(2)} := \sigma_0 \|\varphi_0 * g|L_p(\mathbb{R}^n)\| + \Big(\sum_{j=1}^{\infty} \sigma_j^q \|\varphi_{2^{-j}} * g|L_p(\mathbb{R}^n)\|^q \Big)^{1/q}, \tag{5.3.1}$$

with the usual modification if $q = \infty$*.*

The functions presented next were considered in [FL06].

Notation 5.3.5. *Let* $K \geq -1$ *be an integer. In this section* k_0 *and* k *will denote functions in* $\mathcal{S}(\mathbb{R}^n)$ *such that*

$$|\widehat{k_0}(\xi)| > 0 \quad \textit{for} \quad |\xi| \leq 2,$$

$$|\widehat{k}(\xi)| > 0 \quad \textit{for} \quad \frac{1}{2} \leq |\xi| \leq 2,$$

and

$$\int_{\mathbb{R}^n} x^\alpha k(x)dx = 0 \quad \textit{for any} \quad |\alpha| \le K.$$

We refer to [FL06, pp. 50-51, Lemma 4.4.5] for the proof of the existence of such functions (for each $K \ge 0$).

The following theorem was stated in [FL06, p. 36, Theorem 4.3.4 and Remark 4.3.5].

Theorem 5.3.6. *Let σ be an admissible sequence, $0 < p, q \le \infty$,*

$$K > -1 + \overline{s}(\sigma) \quad \textit{and} \quad r > \frac{n}{p}.$$

Let k_0 and k be functions as in Notation 5.3.5. For all $g \in \mathcal{S}'(\mathbb{R}^n)$

$$\|g|B^\sigma_{p,q}(\mathbb{R}^n)\|^{(1)}_{k_0,k,r} \sim \|g|B^\sigma_{p,q}(\mathbb{R}^n)\| \sim \|g|B^\sigma_{p,q}(\mathbb{R}^n)\|^{(2)}_{k_0,k,r}. \tag{5.3.2}$$

For the next Lemma we refer to [BPT96, p. 224. Lemma 2.1(ii)].

Lemma 5.3.7. *Let m and R be non-negative integers and $0 \le \lambda \le R$. Let $\mu, \nu \in \mathcal{S}(\mathbb{R}^n)$ and assume that ν has moments up to order $m - 1 + R$ that vanish, i.e.,*

$$\int_{\mathbb{R}^n} x^\alpha \nu(x)dx = 0, \quad \textit{for all} \quad \alpha \in {\mathbb{N}_0}^n \quad \textit{with} \quad |\alpha| \le m - 1 + R,$$

with the convention that no moment condition is required when $m - 1 + R < 0$. Then there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} \Big(1 + \frac{|y|}{t}\Big)^\lambda |\nu_t * \mu_s(y)|dy \le C\Big(\frac{t}{s}\Big)^m,$$

for all $0 < t \le s < \infty$, where

$$\nu_t(x) := t^{-n}\nu(t^{-1}x) \quad \textit{and} \quad \mu_s(x) := s^{-n}\mu(s^{-1}x), \quad t, s > 0, \quad x \in \mathbb{R}^n.$$

Definition 5.3.8. *(i) We say that a domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a special Lipschitz domain if there is a Lipschitz function $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ such that*

$$\Omega = \{x = (x', x_n) \in \mathbb{R}^n \ : \ x_n > \omega(x')\}. \tag{5.3.3}$$

(ii) *We say that a domain* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *is a bounded Lipschitz domain if it is a bounded domain whose boundary* $\partial\Omega$ *can be covered by a finite number of open balls* B_j, $j = 1, ..., J$, *centered at points of* $\partial\Omega$, *such that*

$$B_j \cap \Omega = B_j \cap \Omega_j, \quad j = 1, ..., J, \tag{5.3.4}$$

where Ω_j *are rotations of suitable special Lipschitz domains.*

(iii) *A bounded Lipschitz domain in the real line* $\mathbb{R}$ *is the interior of a finite union of disjoint bounded closed intervals.*

Notation 5.3.9. *In what follows if we assume that* Ω *is a special Lipschitz domain in* $\mathbb{R}^n$ *then we will denote by* ω *the Lipschitz function referred in Definition 5.3.8(i), particularly* (5.3.3). *We will denote by* A *a positive constant such that*

$$|\omega(x') - \omega(y')| \leq A|x' - y'|, \quad x', y' \in \mathbb{R}^{n-1}.$$

Moreover we shall write K_A *to denote the cone*

$$K_A := \{(x', x_n) \in \mathbb{R}^n \; : \; |x'| < A^{-1}x_n\},$$

which satisfies the following property

$$x + K_A \subset \Omega, \quad x \in \Omega. \tag{5.3.5}$$

Definition 5.3.10. *Let* Ω *be a special Lipschitz domain in* $\mathbb{R}^n$. *For* $\gamma \in \mathcal{D}(-K_A)$ *and* $f \in \mathcal{D}'(\Omega)$ *we define*

$$\gamma * f(x) :=< f, \gamma(x - \cdot) >, \quad x \in \Omega.$$

Remark 5.3.11. *By* (5.3.5), *Definition 5.3.10 makes sense. In what follows we need to guarantee that the convolution presented in Definition 5.3.10 is associative, namely*

$$(\gamma * \eta) * f(x) = \gamma * (\eta * f)(x), \quad \gamma, \eta \in \mathcal{D}(-K_A), \quad x \in \Omega.$$

This is a consequence of the equality

$$\gamma * f(x) = \gamma * g(x), \quad \gamma \in \mathcal{D}(-K_A), \quad x \in \Omega,$$

which holds for all $f \in \mathcal{D}'(\Omega)$ such that there is $g \in \mathcal{S}'(\mathbb{R}^n)$ with $g_{|\Omega} = f$. Actually, this will be the case in what follows.

The result stated in the next theorem was proved in [Ryc99, pp. 253-255].

Theorem 5.3.12. *Let Ω be a special Lipschitz domain in $\mathbb{R}^n$. There exist four functions η_0, η, θ_0 and θ in $\mathcal{S}(\mathbb{R}^n)$ supported in $-K_A$ such that*

$$\int x^\alpha \eta(x)dx = \int x^\alpha \theta(x)dx = 0, \quad \textit{for all} \quad \alpha \in {\mathbb{N}_0}^n.$$

and

$$f = \theta_0 * \eta_0 * f + \sum_{j=1}^{\infty} \theta_{2^{-j}} * \eta_{2^{-j}} * f \quad \textit{in} \quad \mathcal{D}'(\Omega), \tag{5.3.6}$$

for all $f \in \mathcal{D}'(\Omega)$, where we are using the notation introduced in Notation 5.3.3.

Notation 5.3.13. *For any $g : \Omega \to \mathbb{R}$ we denote by g_Ω its extension from Ω to all $\mathbb{R}^n$ by zero.*

Theorem 5.3.14. *Let Ω be a special Lipschitz domain in $\mathbb{R}^n$. Consider η_0, η, θ_0 and θ as in Theorem 5.3.12. Let f belong to the restriction of $\mathcal{S}'(\mathbb{R}^n)$ to Ω and*

$$\mathscr{E} f := \theta_0 * (\eta_0 * f)_\Omega + \sum_{j=1}^{\infty} \theta_{2^{-j}} * (\eta_{2^{-j}} * f)_\Omega. \tag{5.3.7}$$

Then, for all admissible sequence σ and $0 < p, q \leq \infty$, $\mathscr{E}$ is a linear bounded extension operator from $B^\sigma_{p,q}(\Omega)$ into the corresponding space on $\mathbb{R}^n$.

Proof. Step 1: We fix an admissible sequence σ and $0 < p, q \leq \infty$. We also fix an integer r satisfying $r > n/p$ and an integer K such that

$$K > \max\{-1, \overline{s}(\sigma), -\underline{s}(\sigma)\} + r. \tag{5.3.8}$$

Let us introduce some notation and recall some useful results. Consider, for $f \in \mathcal{D}'(\Omega)$,

$$(\eta_0^* f)^\Omega_r(x) := \sup_{y\in\Omega} \frac{|\eta_0 * f(y)|}{(1+|x-y|)^r}, \quad x \in \mathbb{R}^n,$$

and, for $j \in \mathbb{N}$,

$$(\eta^*_{2^{-j}} f)^\Omega_r(x) := \sup_{y \in \Omega} \frac{|\eta_{2^{-j}} * f(y)|}{(1 + 2^j |x - y|)^r}, \quad x \in \mathbb{R}^n.$$

We define

$$\|f|B^\sigma_{p,q}(\Omega)\|_{\eta_0,\eta,r} := \sigma_0 \|(\eta^*_0 f)^\Omega_r | L_p(\Omega)\| + \Big(\sum_{j=1}^\infty \sigma^q_j \|(\eta^*_{2^{-j}} f)^\Omega_r | L_p(\Omega)\|^q \Big)^{1/q} \tag{5.3.9}$$

with the usual modification if $q = \infty$.

Let $g \in \mathcal{S}'(\mathbb{R}^n)$. Let k_0 and k be functions as in Notation 5.3.5.

There is $c > 0$ independent of g such that

$$\|g|B^\sigma_{p,q}(\mathbb{R}^n)\|^{(1)}_{\eta_0,\eta,r} \le c \|g|B^\sigma_{p,q}(\mathbb{R}^n)\|^{(1)}_{k_0,k,r}. \tag{5.3.10}$$

To prove (5.3.10) we adapt Step 1 of the proof of [FL06, p. 36, Theorem 4.3.4], considering a sequence of functions $(\psi_j)_{j \in \mathbb{N}_0}$ defined by

$$\widehat{\psi_0}(\xi) := \frac{\varphi_0(\xi)}{\widehat{k_0}(\xi)} \quad \text{and} \quad \widehat{\psi_j}(\xi) := \frac{\varphi_j(\xi)}{\widehat{k}(2^{-j}\xi)}, \quad j \in \mathbb{N}, \tag{5.3.11}$$

where $(\varphi_j)_{j \in \mathbb{N}_0}$ is a partition of unity as in Definition 2.1.1 with $N = (2^j)_{j \in \mathbb{N}_0}$ and $l_0 = 1$. It follows from (5.3.11) and the properties of $(\varphi_j)_{j \in \mathbb{N}_0}$ that

$$\eta_{2^{-j}} * g = \eta_{2^{-j}} * \psi_0 * k_0 * g + \sum_{m=1}^\infty \eta_{2^{-j}} * \psi_m * k_{2^{-m}} * g.$$

For $m \ge 1$

$$|\eta_{2^{-j}} * \psi_m * k_{2^{-m}} * g(y)| \le (k^*_{2^{-m}} g)_r(y) I_{j,m},$$

where

$$I_{j,m} := \int_{\mathbb{R}^n} |\eta_{2^{-j}} * \psi_m(z)| \cdot (1 + 2^m |z|)^r dz.$$

Applying the properties of the functions involved and [FL06, p. 33, Lemma 4.3.1] we obtain estimations from above of $I_{j,m}$ concluding that there is $a > 0$ such that for all $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$

$$\sigma_j (\eta^*_{2^{-j}} g)_r(x) \lesssim \sigma_0 (k^*_0 g)_r(x) 2^{-ja} + \sum_{m=1}^\infty \sigma_m (k^*_{2^{-m}} g)_r(x) 2^{-a|j-m|} \tag{5.3.12}$$

and an analogous estimation for $\sigma_0(\eta_0^* g)_r(x)$. By (5.3.12) and applying [FL06, p. 34, Lemma 4.3.2] we conclude (5.3.10).

Step 2: Let $f \in B^\sigma_{p,q}(\Omega)$. In this step we prove that there is $c > 0$ independent of f such that

$$\|f|B^\sigma_{p,q}(\Omega)\|_{\eta_0,\eta,r} \leq c\|f|B^\sigma_{p,q}(\Omega)\|. \tag{5.3.13}$$

By Definition 5.0.4 there is $g \in B^\sigma_{p,q}(\mathbb{R}^n)$ such that $g|_\Omega = f$ in $\mathcal{D}'(\Omega)$ and

$$\|g|B^\sigma_{p,q}(\mathbb{R}^n)\| \sim \|f|B^\sigma_{p,q}(\Omega)\|. \tag{5.3.14}$$

It can be easily verified that, for all $j \in \mathbb{N}$, $\operatorname{supp} \eta_{2^{-j}} \subset -K_A$. So it follows from (5.3.5) that, for all $y \in \Omega$, $\operatorname{supp} \eta_{2^{-j}}(y - \cdot) \subset \Omega$. Therefore, as $g|_\Omega = f$, for all $j \in \mathbb{N}$ and $y \in \Omega$

$$\eta_{2^{-j}} * f(y) =< f, \eta_{2^{-j}}(y - \cdot) >=< g, \eta_{2^{-j}}(y - \cdot) >= \eta_{2^{-j}} * g(y)$$

Hence, for all $x \in \mathbb{R}^n$,

$$(\eta^*_{2^{-j}} f)^\Omega_r(x) \leq (\eta^*_{2^{-j}} g)_r(x).$$

Something analogous can be proved considering η_0 instead of $\eta_{2^{-j}}$. So

$$\|f|B^\sigma_{p,q}(\Omega)\|_{\eta_0,\eta,r} \leq \|g|B^\sigma_{p,q}(\mathbb{R}^n)\|^{(1)}_{\eta_0,\eta,r} \tag{5.3.15}$$

Now (5.3.13) follows from (5.3.2), (5.3.10), (5.3.14) and (5.3.15).

Step 3: We introduce some more notation. Let Y denote the collection of all sequences of measurable functions in $\mathbb{R}^n$, $(v^j)_{j\in\mathbb{N}_0}$, such that

$$\|(v^j)_j\|_Y := \|(\sigma_j V^j)_{j\in\mathbb{N}_0}|\ell_q(L_p)\| \tag{5.3.16}$$

is finite, where, for $j \in \mathbb{N}_0$,

$$V^j(x) := \sup_{y\in\mathbb{R}^n} \frac{|v^j(y)|}{(1 + 2^j|x - y|)^r}, \quad x \in \mathbb{R}^n. \tag{5.3.17}$$

Let us prove that for all $(v^j)_j \in Y$ such that $\theta_0 * v^0$ and $\theta_{2^{-j}} * v^j$, $j \in \mathbb{N}$, makes sense,

$$\theta_0 * v^0 + \sum_{j=1}^{\infty} \theta_{2^{-j}} * v^j \tag{5.3.18}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\left\| \theta_0 * v^0 + \sum_{j=1}^{\infty} \theta_{2^{-j}} * v^j \middle| B^{\sigma}_{p,q}(\mathbb{R}^n) \right\| \lesssim \|(v^j)_j\|_Y. \tag{5.3.19}$$

Let $l, j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Then

$$\sigma_l |k_{2^{-l}} * \theta_{2^{-j}} * v^j(x)| \le \sigma_l \, V^j(x) \, \mathcal{I}_{j,l},$$

where

$$\mathcal{I}_{j,l} := \int_{\mathbb{R}^n} |k_{2^{-l}} * \theta_{2^{-j}}(y)|(1 + 2^j|y|)^r dy.$$

Now we apply [BPT96, p. 224. Lemma 2.1(ii)] to estimate $\mathcal{I}_{j,l}$. Let m be a non-negative integer such that

$$\max\{-\underline{s}(\sigma), \overline{s}(\sigma)\} < m \le K + 1 - r.$$

It follows from (5.3.8) that exists such an m.

Let us consider first the case $j \ge l$. We apply Lemma 5.3.7, considering

$$t = 2^{-j}, \quad s = 2^{-l}, \quad \mu = k, \quad \nu = \theta, \quad \lambda = R = r.$$

Hence there is c such that

$$\mathcal{I}_{j,l} \le c 2^{-(j-l)m}.$$

Now we assume that $j < l$. and apply Lemma 5.3.7, considering

$$t = 2^{-l}, \quad s = 2^{-j}, \quad \mu = \theta, \quad \nu = k, \quad \lambda = R = r.$$

So there is c' such that

$$\mathcal{I}_{j,l} \le c' 2^{-(l-j)m}.$$

Therefore, considering $\delta \in (0, m - \max\{-\underline{s}(\sigma), \overline{s}(\sigma)\})$, we get to

$$\begin{aligned} \sigma_l |k_{2^{-l}} * \theta_{2^{-j}} * v^j(x)| &\lesssim \sigma_l \, V^j(x) \, 2^{-|j-l|m} \\ &\lesssim \sigma_j \, V^j(x) \, 2^{-|j-l|(m+\min\{\underline{s}(\sigma)-\delta, -\overline{s}(\sigma)-\delta\})} \\ &= \sigma_j \, V^j(x) \, 2^{-|j-l|\rho}, \end{aligned} \tag{5.3.20}$$

for some $\rho > 0$. Analogously one can prove that for all $m \in \mathbb{N}$

$$\sigma_l |k_{2^{-l}} * \theta_0 * v^0(x)| \lesssim \sigma_l\, V^0(x) 2^{-lm} \lesssim \sigma_0 V^0(x) 2^{-l\rho}, \tag{5.3.21}$$

that for all $j \in \mathbb{N}$

$$\sigma_0 |k_0 * \theta_{2^{-j}} * v^j(x)| \lesssim \sigma_0\, V^j(x) 2^{-jm} \lesssim \sigma_j V^j(x) 2^{-j\rho} \tag{5.3.22}$$

and that

$$\sigma_0 |k_0 * \theta_0 * v^0(x)| \lesssim \sigma_0 V^0(x). \tag{5.3.23}$$

By (5.3.2), (5.3.16)-(5.3.17), (5.3.20) and (5.3.22), for all $j \in \mathbb{N}$

$$\begin{aligned} \|\theta_{2^{-j}} * v^j | B^{(-2\rho)\sigma}_{p,q}\| &\sim \sigma_0 \|k_0 * \theta_{2^{-j}} * v^j | L_p(\mathbb{R}^n)\| + \Big(\sum_{l=1}^{\infty} 2^{-2l\rho q} \sigma_l^q \|k_{2^{-l}} * \theta_{2^{-j}} * v^j | L_p(\mathbb{R}^n)\|^q \Big)^{1/q} \\ &\lesssim 2^{-j\rho} \|(v^l)_l\|_Y. \end{aligned} \tag{5.3.24}$$

with the usual modification if $q = \infty$. Analogously one estimates $\|\theta_0 * v^0 | B^{(-2\rho)\sigma}_{p,q}\|$. Letting

$$t = \min\{1, p, q\} \quad \text{and} \quad F_T := \theta_0 * v^0 + \sum_{j=1}^{T} \theta_{2^{-j}} * v^j$$

one obtains, for $T_1, T_2 \in \mathbb{N}$, $T_2 > T_1$,

$$\|F_{T_1} - F_{T_2} | B^{(-2\rho)\sigma}_{p,q}(\mathbb{R}^n)\| \le \|(v^l)_l\|_Y \Big(\sum_{j=T_1+1}^{T_2} 2^{-j\rho t} \Big)^{1/t}.$$

Therefore $(F_T)_T$ is a Cauchy sequence in the complete space $B^{(-2\rho)\sigma}_{p,q}(\mathbb{R}^n)$ and, so, convergent in this space. As $B^{(-2\rho)\sigma}_{p,q}(\mathbb{R}^n)$ is continuously embedded in $\mathcal{S}'(\mathbb{R}^n)$, $(F_T)_T$ also converges in $\mathcal{S}'(\mathbb{R}^n)$ to the same limit. Thus the expression in (5.3.18) belongs to $\mathcal{S}'(\mathbb{R}^n)$.

Now let us prove (5.3.19). Consider $\widetilde{p} := \min(1, p)$. It follows from (5.3.20)-(5.3.23) that

$$\Big\| \sigma_l k_{2^{-l}} * (\theta_0 * v^0 + \sum_{j=1}^{\infty} \theta_{2^{-j}} * v^j) | L_p(\mathbb{R}^n) \Big\|^{\widetilde{p}} \lesssim \sum_{j=0}^{\infty} \Big\| \sigma_j V^j 2^{-|l-j|\rho} | L_p(\mathbb{R}^n) \Big\|^{\widetilde{p}}, \tag{5.3.25}$$

for $l \in \mathbb{N}$ and

$$\Big\| \sigma_0 k_0 * (\theta_0 * v^0 + \sum_{j=1}^{\infty} \theta_{2^{-j}} * v^j) | L_p(\mathbb{R}^n) \Big\|^{\widetilde{p}} \lesssim \sum_{j=0}^{\infty} \Big\| \sigma_j V^j 2^{-j\rho} | L_p(\mathbb{R}^n) \Big\|^{\widetilde{p}}. \tag{5.3.26}$$

So, by (5.3.25)-(5.3.26) and following the notation in (5.3.1), for $0 < q < \infty$,

$$\begin{aligned}
\Bigg(\Big\|\theta_0 * v^0 + \sum_{j=1}^{\infty} \theta_{2^{-j}} * v^j | B^\sigma_{p,q}(\mathbb{R}^n)\Big\|^{(2)}_{k_0,k,r}\Bigg)^q &\lesssim \sum_{l=0}^{\infty}\Big(\sum_{j=0}^{\infty}\|\sigma_j V^j 2^{-|l-j|\rho}|L_p(\mathbb{R}^n)\|^{\widetilde{p}}\Big)^{\frac{q}{\widetilde{p}}} \\
&\lesssim \sum_{l=0}^{\infty} 2^{-l\rho q}\Big(\sum_{j=0}^{l} 2^{j\rho\widetilde{p}}\|\sigma_j V^j|L_p(\mathbb{R}^n)\|^{\widetilde{p}}\Big)^{\frac{q}{\widetilde{p}}} \\
&\quad + \sum_{l=0}^{\infty} 2^{l\rho q}\Big(\sum_{j=l+1}^{\infty} 2^{-j\rho\widetilde{p}}\|\sigma_j V^j|L_p(\mathbb{R}^n)\|^{\widetilde{p}}\Big)^{\frac{q}{\widetilde{p}}} \\
&\lesssim \sum_{j=0}^{\infty}\|\sigma_j V^j|L_p(\mathbb{R}^n)\|^q \qquad (5.3.27) \\
&= \|(v^j)_j\|_Y^q, \qquad (5.3.28)
\end{aligned}$$

where we applied in (5.3.27) a Hardy inequality which was stated in Lemma 3.4.16. For $q = \infty$ an analogous estimate can be obtained with the usual modifications. Now (5.3.19) is a consequence of Theorem 5.3.6 and (5.3.28).

Step 4: In this step we fix

$$v^0 := (\eta_0 * f)_\Omega \quad \text{and} \quad v^j := (\eta_{2^{-j}} * f)_\Omega, \quad j \in \mathbb{N}, \tag{5.3.29}$$

where f is as in Step 2, i.e., $f \in B^\sigma_{p,q}(\Omega)$ and where we are using the notation presented in Notation 5.3.13. We remark that, by what was done previously and by [Kho72, Tome I, p. 222, Theorem], $(\eta_0 * f)_\Omega$ and $(\eta_{2^{-j}} * f)_\Omega$, $j \in \mathbb{N}$, are locally integrable. Therefore, by [Kho72, Tome I, p. 121, Theorem], $\theta_0 * (\eta_0 * f)_\Omega$ and $\theta_{2^{-j}} * (\eta_{2^{-j}} * f)_\Omega$, $j \in \mathbb{N}$, are well-defined. We prove that

$$\|(v^j)_j\|_Y \lesssim \|f|B^\sigma_{p,q}(\Omega)\|_{\eta_0,\eta,r}. \tag{5.3.30}$$

By (5.3.9) and (5.3.16), it is enough to prove that

$$\|V^0|L_p(\mathbb{R}^n)\| \lesssim \|(\eta_0^* f)_r^\Omega|L_p(\Omega)\| \quad \text{and} \quad \|V^j|L_p(\mathbb{R}^n)\| \lesssim \|(\eta_{2^{-j}}^* f)_r^\Omega|L_p(\Omega)\|, \quad j \in \mathbb{N}. \tag{5.3.31}$$

We define the application $S : \mathbb{R}^n \setminus \overline{\Omega} \to \Omega$ by

$$S(x) := (x', 2\omega(x') - x_n), \quad \text{where} \quad x = (x', x_n), \quad x' \in \mathbb{R}^{n-1}, \quad x_n \le \omega(x').$$

We remark that $S(x)$, with $x \in \mathbb{R}^n \setminus \overline{\Omega}$, is the symmetric point of x with respect to $\partial\Omega$ and that there is a number $B > 0$ such that

$$|S(x) - y| \leq B|x - y|, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}, \quad y \in \Omega. \tag{5.3.32}$$

Let $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$. Then

$$V^j(x) = \sup_{y \in \mathbb{R}^n} \frac{|(\eta_{2^{-j}} * f)_\Omega(y)|}{(1 + 2^j|x - y|)^r} = \sup_{y \in \Omega} \frac{|\eta_{2^{-j}} * f(y)|}{(1 + 2^j|x - y|)^r}.$$

So, if $x \in \Omega$, $V^j(x) = (\eta^*_{2^{-j}} f)^\Omega_r(x)$. If $x \in \mathbb{R}^n \setminus \overline{\Omega}$, applying (5.3.32), we get to

$$V^j(x) = \sup_{y \in \Omega} \frac{|\eta_{2^{-j}} * f(y)|}{(1 + 2^j|x - y|)^r} \lesssim \sup_{y \in \Omega} \frac{|\eta_{2^{-j}} * f(y)|}{(1 + 2^j|S(x) - y|)^r} = (\eta^*_{2^{-j}} f)^\Omega_r(S(x)).$$

Therefore, for $j \in \mathbb{N}$,

$$\|V^j|L_p(\mathbb{R}^n)\|^p \lesssim \int_\Omega |(\eta^*_{2^{-j}} f)^\Omega_r(x)|^p dx + \int_{\mathbb{R}^n \setminus \overline{\Omega}} |(\eta^*_{2^{-j}} f)^\Omega_r(S(x))|^p dx \tag{5.3.33}$$

and an analogous estimate can be obtained for $\|V^0|L_p(\mathbb{R}^n)\|$ with η_0 instead of $\eta_{2^{-j}}$.

Given a set D in $\mathbb{R}^n$ let χ_D denote the characteristic function with respect D. Let us prove that given a function $A : \mathbb{R}^n \to \mathbb{C}$ such that $\chi_\Omega A \in L_1(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n \setminus \overline{\Omega}} A(S(x)) dx = \int_\Omega A(x) dx. \tag{5.3.34}$$

For such a function A, applying Fubini, we obtain

$$\begin{aligned} \int_\Omega A(x)dx &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_\Omega(x', y_n) A(x', y_n) dy_n dx' \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_{\mathbb{R}^n \setminus \overline{\Omega}}(x', 2\omega(x') - y_n) A(x', y_n) dy_n dx' \qquad (5.3.35) \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_{\mathbb{R}^n \setminus \overline{\Omega}}(x', x_n) A(x', 2\omega(x') - x_n) dx_n dx' \\ &= \int_{\mathbb{R}^n \setminus \overline{\Omega}} A(S(x)) dx, \end{aligned}$$

where (5.3.35) is justified by

$$\chi_{\mathbb{R}^n \setminus \overline{\Omega}}(x', 2\omega(x') - y_n) = \chi_\Omega(x', y_n), \quad x' \in \mathbb{R}^{n-1}, \quad y_n = 2\omega(x') - x_n, \tag{5.3.36}$$

which follows from (5.3.3).

Hence (5.3.31) follows from (5.3.33) and (5.3.34) taking $A = |(\eta_0^* f)_r^\Omega|^p$ and $A = |(\eta_{2^{-j}}^* f)_r^\Omega|^p$, respectively.

Step 5: The conclusion can be obtained immediately from the previous steps as follows: let $f \in B^\sigma_{p,q}(\Omega)$. For $(v^j)_j$ as in (5.3.29), the expression in (5.3.18) coincides with $\mathscr{E} f$. By (5.3.13) and (5.3.30)

$$\|(v^j)_j\|_Y \lesssim \|f|B^\sigma_{p,q}(\Omega)\|.$$

By (5.3.19)

$$\|\mathscr{E} f|B^\sigma_{p,q}(\mathbb{R}^n)\| \lesssim \|(v^j)_j\|_Y.$$

Finally, (5.3.5), the fact that $\operatorname{supp}\theta_0, \operatorname{supp}\theta \subset -K_A$, (5.3.6) and (5.3.7) imply that $(\mathscr{E} f)|_\Omega = f$. □

Theorem 5.3.15. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. There is a universal extension operator, i.e., there is an extension operator $\mathscr{E}$ such that, for all admissible sequences σ and $0 < p, q \leq \infty$,*

$$\mathscr{E} : B^\sigma_{p,q}(\Omega) \to B^\sigma_{p,q}(\mathbb{R}^n).$$

Proof. Let J, B_j and Ω_j, $j = 1, ..., J$ be as in Definition 5.3.8 (ii). For simplicity let us assume that Ω_j are special Lipschitz domains (without the need to consider rotations). By Theorem 5.3.14, for every $j = 1, ..., J$, there is a universal extension operator, which we denote by $\mathscr{E}_j$, acting from Besov spaces on Ω_j into the corresponding Besov spaces on $\mathbb{R}^n$. We consider a partition of the unity on Ω as follows: let $\phi \in \mathcal{D}(\Omega)$ and $\phi_j \in \mathcal{D}(B_j)$ such that

$$\phi + \sum_{j=1}^J \phi_j = 1 \quad \text{on} \quad \Omega.$$

We consider $\delta > 0$ such that the closure of the neighborhood of $\operatorname{supp}\phi$, $(\operatorname{supp}\phi)_\delta$ is a subset of Ω. Analogously, for every $j = 1, ..., J$ we consider $\delta_j > 0$ such that the closure of the neighborhood of $\operatorname{supp}\phi_j$, $(\operatorname{supp}\phi_j)_{\delta_j}$, is a subset of B_j. We also fix auxiliary smooth functions ψ and ψ_j, $j, = 1, ..., J$, which satisfy

$$\psi(x) = 1 \quad \text{if} \quad x \in \operatorname{supp}\phi \qquad \text{and} \qquad \psi_j(x) = 0 \quad \text{if} \quad x \notin (\operatorname{supp}\phi)_\delta \tag{5.3.37}$$

and

$$\psi_j(x) = 1 \quad \text{if} \quad x \in \operatorname{supp}\phi_j \qquad \text{and} \qquad \psi_j(x) = 0 \quad \text{if} \quad x \notin (\operatorname{supp}\phi_j)_{\delta_j}. \tag{5.3.38}$$

We fix an admissible sequence σ and $0 < p, q \leq \infty$. Let $f \in B^\sigma_{p,q}(\Omega)$ and $g \in B^\sigma_{p,q}(\mathbb{R}^n)$ be such that

$$g_{|\Omega} = f \quad \text{and} \quad \|f|B^\sigma_{p,q}(\Omega)\| \sim \|g|B^\sigma_{p,q}(\mathbb{R}^n)\|. \tag{5.3.39}$$

We define, for $j = 1, ..., J$, the auxiliary distributions:

$$< f^j, \varphi >:=< f, \phi_j\varphi >, \quad \varphi \in \mathcal{D}(\Omega_j). \tag{5.3.40}$$

Using (5.3.4) one can easily prove that

$$f^j \in \mathcal{D}'(\Omega_j) \quad \text{and} \quad (g\phi_j)_{|\Omega_j} = f^j.$$

Applying Theorem 5.2.1 one concludes that $f^j \in B^\sigma_{p,q}(\Omega_j)$ and there exist c_j, $j = 1, ..., J$, such that

$$\|f^j|B^\sigma_{p,q}(\Omega_j)\| \leq \|g\phi_j|B^\sigma_{p,q}(\mathbb{R}^n)\| \leq c_j\|g|B^\sigma_{p,q}(\mathbb{R}^n)\| \lesssim \|f|B^\sigma_{p,q}(\Omega)\|, \tag{5.3.41}$$

where we also applied (5.0.3) and (5.3.39). We define

$$\mathscr{E}f := (f\phi)_\Omega + \sum_{j=1}^{J} \psi_j(\mathscr{E}_j f^j), \tag{5.3.42}$$

where $(f\phi)_\Omega$ is defined by

$$< (f\phi)_\Omega, \varphi >=< f\phi, (\varphi\psi)_{|\Omega} >, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Let us prove first that

$$\mathscr{E}f \in B^\sigma_{p,q}(\mathbb{R}^n) \quad \text{and} \quad \|\mathscr{E}f|B^\sigma_{p,q}(\mathbb{R}^n)\| \lesssim \|f|B^\sigma_{p,q}(\Omega)\|. \tag{5.3.43}$$

Clearly the restriction of $(f\phi)_\Omega$ to Ω coincides with $f\phi$. Let us prove that

$$\|(f\phi)_\Omega|B^\sigma_{p,q}(\mathbb{R}^n)\| \lesssim \|f|B^\sigma_{p,q}(\Omega)\|. \tag{5.3.44}$$

Let $v \in B^\sigma_{p,q}(\mathbb{R}^n)$ be such that $v_{|\Omega} = f$. Then $v\phi$ coincides with $(f\phi)_\Omega$ and the restriction of $v\phi$ to Ω coincides with $f\phi$. Therefore, also applying Theorem 5.2.1, one obtains

$$\|(f\phi)_\Omega|B^\sigma_{p,q}(\mathbb{R}^n)\| = \|v\phi|B^\sigma_{p,q}(\mathbb{R}^n)\| \le c\, c_\phi \|v|B^\sigma_{p,q}(\mathbb{R}^n)\|. \tag{5.3.45}$$

By Definition 5.0.4(i) and considering in (5.3.45) the infimum of all $v \in B^\sigma_{p,q}(\mathbb{R}^n)$ such that $v_{|\Omega} = f$ one concludes

$$\|(f\phi)_\Omega|B^\sigma_{p,q}(\mathbb{R}^n)\| \lesssim \|f|B^\sigma_{p,q}(\Omega)\|. \tag{5.3.46}$$

By Theorem 5.3.14, $\mathscr{E}_j f^j \in B^\sigma_{p,q}(\mathbb{R}^n)$ and, by Theorem 5.2.1, $\psi_j(\mathscr{E}_j f^j) \in B^\sigma_{p,q}(\mathbb{R}^n)$. Moreover there exist d_j, $j = 1, ..., J$, such that

$$\|\psi_j(\mathscr{E}_j f^j)|B^\sigma_{p,q}(\mathbb{R}^n)\| \le d_j \|\mathscr{E}_j f^j|B^\sigma_{p,q}(\mathbb{R}^n)\| \lesssim d_j \|f^j|B^\sigma_{p,q}(\Omega_j)\|. \tag{5.3.47}$$

Now (5.3.43) follows from (5.3.41), (5.3.42) and (5.3.47).
It remains to prove that $(\mathscr{E} f)_{|\Omega} = f$. Let $\varphi \in \mathcal{D}(\Omega)$. For all $j = 1, ..., J$, $\psi_j\varphi$ belongs to $\mathcal{D}(B_j \cap \Omega)$, which is, by (5.3.4), a subset of $\mathcal{D}(\Omega_j)$. Therefore

$$< \psi_j(\mathscr{E}_j f^j), \varphi > = < \mathscr{E}_j f^j, \psi_j\varphi > = < f^j, \psi_j\varphi > . \tag{5.3.48}$$

From (5.3.48), applying first (5.3.40) and then (5.3.38), one gets

$$< \psi_j(\mathscr{E}_j f^j), \varphi > = < f, \phi_j\psi_j\varphi > = < f, \phi_j\varphi > = < f\phi_j, \varphi > .$$

Hence

$$< \mathscr{E} f, \varphi > = < f\phi + \sum_{j=1}^J f\phi_j, \varphi > = < f, \varphi > \tag{5.3.49}$$

and the proof is concluded. $\square$

5.4 The Laplacian

In this section Ω denotes a bounded C^∞ domain in $\mathbb{R}^n$. For the definition of bounded C^∞ domain we refer to [Tri83, 3.2.1, p. 191], for example.

As usual

$$-\Delta := -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is the Laplacian in Ω.

We use the notation *Dirichlet Laplacian* always with the understanding that the vanishing boundary data at $\partial\Omega$ are incorporated in the definition of the domain of $-\Delta$ in the function spaces considered, as we shall see in this section.

In this section we study the Dirichlet Laplacian acting in Besov spaces of generalised smoothness on bounded smooth domains.

First we recall a well-known result for this operator acting in the classical Besov spaces.

Theorem 5.4.1. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$,*

$$1 \le p \le \infty, \quad 1 \le q \le \infty \quad \text{and} \quad s > \frac{1}{p}.$$

Then $-\Delta$ maps

$$\{g \in B^{(s)}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega}\, g = 0\} \quad \text{isomorphically onto} \quad B^{(s-2)}_{p,q}(\Omega).$$

This result can be found in [Tri01, p. 255] where the following references are given: [Tri95, Remark 1, 5.7.1], [Tri83, 4.3.3, 4.3.4] and [RS96, 3.5.2, p. 130]. Our intention in this section is to extend this result to spaces of generalised smoothness applying interpolation. For this purpose it is convenient to prove that (5.1.3) holds if one considers Besov spaces on some class of domains, instead of $\mathbb{R}^n$. For that we will use the extension operator referred in Theorem 5.3.15. In [Tri06, p. 69, Theorem 1.110] this extension operator was used to get interpolation results for function spaces on Lipschitz domains from corresponding results on $\mathbb{R}^n$, for the real interpolation method for quasi-Banach spaces and the classical complex interpolation method for Banach spaces, always in the context of classic Besov spaces. For real interpolation with a function parameter the result can be obtained by the same method.

Proposition 5.4.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let σ be an admissible sequence, $0 < p \leq \infty$ and $0 < q_0, q_1, q \leq \infty$. Let $s_0, s_1 \in \mathbb{R}$ satisfy $s_1 < \underline{s}(\sigma) \leq \overline{s}(\sigma) < s_0$ and γ be given by* (5.1.2)*. Then*

$$\left(B^{(s_0)}_{p,q_0}(\Omega), B^{(s_1)}_{p,q_1}(\Omega)\right)_{\gamma,q} = B^{\sigma}_{p,q}(\Omega).$$

Proof. Let us temporarily use the following notation:

$$A_i = B^{(s_i)}_{p,q_i}(\mathbb{R}^n) \qquad \text{and} \qquad B_i = B^{(s_i)}_{p,q_i}(\Omega), \quad i = 0, 1.$$

Then

$$A_0 \subset A_1 \subset \mathcal{S}'(\mathbb{R}^n) \qquad \text{and} \qquad B_0 \subset B_1 \subset \mathcal{D}'(\Omega).$$

Let us denote by $\mathscr{R}$ the restriction to $\mathcal{D}'(\Omega)$ and by $\mathscr{E}$ the extension operator referred in Theorem 5.3.15. As $\mathscr{R}\mathscr{E}$ is the identity operator in B_i, id_{B_i}, then, using the notions of interpolation (some of them also notions of the theory of categories), $\mathscr{R}$ is a retraction in $L(A_i, B_i)$ and $\mathscr{E}$ is a coretraction belonging to $\mathscr{R}$. We say that B_i is a retract of A_i. For a systematic treatment of the interpolation theory in Banach spaces, including the notions and results we are applying, we refer to [Tri95, Chapter 1]. For a collection of notions and results on real interpolation with function parameter of quasi-Banach spaces we refer to [Alm05a, Chapter 2].

Applying Definition 1.2.18 and Proposition 1.2.20, one can verify that, under the conditions considered, $0 < \underline{S}(\gamma) \leq \overline{S}(\gamma) < 1$. So one can apply Theorem 2.2.6 and Remark 2.2.7 in [Alm05a, pp. 20-21] to conclude that $(B_0, B_1)_{\gamma,q}$ is a retract of $(A_0, A_1)_{\gamma,q}$ (with "the same mappings" $\mathscr{R}$ and $\mathscr{E}$) and

$$\|f|(B_0, B_1)_{\gamma,q}\| \sim \|\mathscr{E}f|(A_0, A_1)_{\gamma,q}\|, \quad f \in (B_0, B_1)_{\gamma,q}.$$

Let $f \in (B_0, B_1)_{\gamma,q}$. Then, by the above mentioned results,

$$\mathscr{E}f \in (A_0, A_1)_{\gamma,q} = B^{\sigma}_{p,q}(\mathbb{R}^n) \qquad \text{and} \qquad \|\mathscr{E}f|B^{\sigma}_{p,q}(\mathbb{R}^n)\| \sim \|f|(B_0, B_1)_{\gamma,q}\|.$$

Using the fact that $(B_0, B_1)_{\gamma,q}$ is a retract of $(A_0, A_1)_{\gamma,q}$ and Definition 5.0.4, we conclude that

$$f = \mathscr{R}\mathscr{E}f \in B^{\sigma}_{p,q}(\Omega) \qquad \text{and} \qquad \|f|B^{\sigma}_{p,q}(\Omega)\| \lesssim \|f|(B_0, B_1)_{\gamma,q}\|.$$

We prove now the reverse inclusion. Let $f \in B^{\sigma}_{p,q}(\Omega)$. Then, by Theorem 5.3.15,

$$\mathscr{E}f \in B^{\sigma}_{p,q}(\mathbb{R}^n) \qquad \text{and} \qquad \|\mathscr{E}f|B^{\sigma}_{p,q}(\mathbb{R}^n)\| \lesssim \|f|B^{\sigma}_{p,q}(\Omega)\|. \tag{5.4.1}$$

Again by [Alm05a, p. 20, Theorem 2.2.6] one concludes that $f = \mathscr{R}\mathscr{E}f \in (B_0, B_1)_{\gamma,q}$ and

$$\|f|(B_0, B_1)_{\gamma,q}\| \lesssim \|\mathscr{E}f|(A_0, A_1)_{\gamma,q}\|$$

and so, by (5.4.1), we conclude.

□

Proposition 5.4.3. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $0 < p \le \infty$ and $0 < q \le \infty$. Let σ be an admissible sequence and $s_0, s_1 \in \mathbb{R}$ such that*

$$\frac{1}{p} + \max\Big(0, (n-1)\big(\frac{1}{p} - 1\big)\Big) < s_1 < \underline{s}(\sigma) \le \overline{s}(\sigma) < s_0$$

and γ be given by (5.1.2). Then

$$\big(\{f \in B^{(s_0)}_{p,q_0}(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\}, \{f \in B^{(s_1)}_{p,q_1}(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\}\big)_{\gamma,q} = \{f \in B^{\sigma}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\}. \tag{5.4.2}$$

Proof. Let us temporarily use the following notation

$$A_i = B^{(s_i)}_{p,q_i}(\Omega), \quad B_i = \{f \in A_i : \mathrm{tr}_{\partial\Omega} f = 0\} \quad \text{and} \quad B = \{f \in A_0 + A_1 : \mathrm{tr}_{\partial\Omega} f = 0\}.$$

To prove this result we will use the fact that B is the range of a projection in $A_0 + A_1$. We recall that we say that Q is a projection in a subspace, A, of a quasi-Banach space if it is a linear operator acting in A such that $Q^2 = Q$ in A. As $s_1 < s_0$ one can prove, applying Proposition 2.1.5, that $A_0 + A_1 = A_1$ (equivalent quasi-norms). Therefore

$$B = \{f \in A_1 : \mathrm{tr}_{\partial\Omega} = 0\}.$$

By [Tri83, p. 200, Theorem 3.3.3] the operator $\mathcal{R}f := \mathrm{tr}_{\partial\Omega} f$ is a retraction from $B^{(s_1)}_{p,q}(\Omega)$ onto $B^{s_1-\frac{1}{p}}_{p,q}(\partial\Omega)$. For the definition of these spaces on $\partial\Omega$, which will not be applied directly in this proof, we refer to [Tri83, p. 192, 3.2.2]. We mean by retraction that $\mathcal{R}$ is a continuous linear map from from $B^{(s_1)}_{p,q}(\Omega)$ onto $B^{s_1-\frac{1}{p}}_{p,q}(\partial\Omega)$ and there exists a continuous linear map, say $\mathcal{G}$, from $B^{s_1-\frac{1}{p}}_{p,q}(\partial\Omega)$ into $B^{(s_1)}_{p,q}(\Omega)$ such that $\mathcal{R}\mathcal{G} = E$ is the identity in $B^{s_1-\frac{1}{p}}_{p,q}(\partial\Omega)$. Let

$$P = \mathcal{G}\mathcal{R} \qquad \text{and} \qquad Q = I - P,$$

where I stands for the identity operator in $B^{(s_1)}_{p,q}(\Omega)$. Both P and Q are projections in $B^{(s_1)}_{p,q}(\Omega)$, i.e., $P^2 = P$ and $Q^2 = Q$. Moreover

$$QB^{(s_1)}_{p,q}(\Omega) = \{f \in B^{(s_1)}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\} = B. \tag{5.4.3}$$

So let us prove (5.4.2). The inclusion "$\subseteq$" is immediate since $B_i \subset A_i$, $i = 0, 1$. Let us prove the reverse inclusion. Let $f \in B^{\sigma}_{p,q}(\Omega)$ be such that $\mathrm{tr}_{\partial\Omega} f = 0$.By Proposition 5.4.2, $f \in (A_0, A_1)_{\gamma,q}$ and $\|f|(A_0, A_1)_{\gamma,q}\| \lesssim \|f|B^{\sigma}_{p,q}(\Omega)\|$. Using the equality $A_0 + A_1 = A_1$ one concludes that $f \in B_0 + B_1$. Let us prove that

$$\|f|(B_0, B_1)_{\gamma,q}\| \lesssim \|f|(A_0, A_1)_{\gamma,q}\|,$$

proving that there is $c > 0$ such that, for all $t > 0$,

$$K(t, f, A_0, A_1) \leq cK(t, f, B_0, B_1), \tag{5.4.4}$$

where K is as in (5.1.1). Let $a_i \in A_i$ be such that $f = a_0 + a_1$. As $f \in B$ then it follows from (5.4.3) that $Qf = f$. Let as consider $b_i = Qa_i$, $i = 0, 1$. Hence

$$b_0 + b_1 = Q(a_0 + a_1) = Qf = f \quad \text{and} \quad \|b_i|B_i\| \lesssim \|a_i|A_i\|, \quad i = 0, 1,$$

which implies that (5.4.4) is satisfied, concluding the proof. $\square$

Theorem 5.4.4. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$,*

$$1 \leq p \leq \infty, \quad 1 \leq q \leq \infty$$

and σ be an admissible sequence such that

$$\underline{s}(\sigma) > \frac{1}{p}.$$

Then $-\Delta$ maps

$$\{g \in B^{\sigma}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega}\, g = 0\} \quad \textit{isomorphically onto} \quad B^{\sigma(-2)}_{p,q}(\Omega).$$

Proof. Let $\phi \in \mathbf{B}_\sigma$. Let $s_0, s_1 \in \mathbb{R}$ be such that

$$\frac{1}{p} < s_1 < \underline{s}(\sigma) \le \overline{s}(\sigma) < s_0$$

and γ be as in (5.1.2). Then, by Proposition 5.4.3, (5.4.2) holds. The function given by $\psi(t) = t^{-2}\phi(t)$ belongs to $\mathbf{B}_{\sigma(-2)}$. Moreover,

$$s_1 - 2 < \underline{s}(\sigma) - 2 = \underline{s}(\sigma(-2)) \le \overline{s}(\sigma(-2)) = \overline{s}(\sigma) - 2 < s_0 - 2$$

and for all $t > 0$

$$\gamma(t) = \frac{t^{\frac{s_0-2}{(s_0-2)-(s_1-2)}}}{\psi\Big(t^{\frac{1}{(s_0-2)-(s_1-2)}}\Big)}.$$

So, by Proposition 5.4.2,

$$\big(B^{(s_0-2)}_{p,q_0}(\Omega), B^{(s_1-2)}_{p,q_1}(\Omega)\big)_{\gamma,q} = B^{\sigma(-2)}_{p,q}(\Omega). \tag{5.4.5}$$

By Theorem 5.4.1, for $i = 0, 1$, $-\Delta$ maps

$$\{g \in B^{(s_i)}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega}\, g = 0\} \quad \text{isomorphically onto} \quad B^{(s_i-2)}_{p,q}(\Omega), \tag{5.4.6}$$

therefore, using the interpolation property, the result follows from (5.4.2), (5.4.5) and (5.4.6). $\square$

5.5 Complements on h-sets: an identification operator

In this section we present an operator which identifies an element of $L_p(\Gamma)$, where Γ is an h-set in $\mathbb{R}^n$ and $1 \le p \le \infty$, with a tempered distribution. We collect some results on this operator proved by Bricchi and we relate it with the trace operator treated in Chapter 3.

Definition 5.5.1. *Let* $1 \leq p \leq \infty$ *and* Γ *be an* h*-set in* $\mathbb{R}^n$*. We define*

$$\mathrm{id}^\Gamma : L_p(\Gamma) \to \mathcal{S}'(\mathbb{R}^n) \tag{5.5.1}$$

by

$$< \mathrm{id}^\Gamma f, \varphi >:= \int_\Gamma f(\gamma)(\varphi_{|\Gamma})(\gamma) d\mu(\gamma), \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{5.5.2}$$

Definition 5.5.2. *Let* Γ *be an* h*-set in* $\mathbb{R}^n$, σ *be an admissible sequence and* $0 < p, q \leq \infty$*. Then we define*

$$B^{\sigma,\Gamma}_{p,q}(\mathbb{R}^n) := \{f \in B^\sigma_{p,q}(\mathbb{R}^n) :< f, \varphi >= 0 \textit{ if } \varphi \in \mathcal{S}(\mathbb{R}^n),\ \varphi_{|\Gamma} = 0\}.$$

The following result was proved in [Bri01, p. 92, Theorem 3.2.8].

Proposition 5.5.3. *Let* Γ *be an* h*-set in* $\mathbb{R}^n$, $1 \leq p \leq \infty$ *and* p' *denote the conjugate exponent of* p*. Then*

$$\mathrm{id}^\Gamma : L_p(\Gamma) \to B^{\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}},\Gamma}_{p,\infty}(\mathbb{R}^n). \tag{5.5.3}$$

If, additionally, $1 < p \leq \infty$ *and* Γ *satisfies the ball condition, then*

$$\mathrm{id}^\Gamma L_p(\Gamma) = B^{\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}},\Gamma}_{p,\infty}(\mathbb{R}^n).$$

The next result follows immediately from Propositions 3.2.2 and 5.5.3.

Proposition 5.5.4. *Let* Γ *be an* h*-set in* $\mathbb{R}^n$, $1 \leq p \leq \infty$ *and* p' *denote the conjugate exponent of* p*. Let*

$$\mathrm{tr}^\Gamma := \mathrm{id}^\Gamma \circ \mathrm{tr}_\Gamma .$$

Then

$$\mathrm{tr}^\Gamma : B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,1}(\mathbb{R}^n) \to B^{\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}_{p,\infty}(\mathbb{R}^n).$$

Let $1 \le p < \infty$ and $g \in (L_p(\Gamma))'$. There is a uniquely determined $g^* \in L_{p'}(\Gamma)$ such that

$$< g, f >= \int_\Gamma f(\gamma) g^*(\gamma) d\mu(\gamma), \quad \text{for all} \quad f \in L_p(\Gamma), \tag{5.5.4}$$

where

$$\|g|(L_p(\Gamma))'\| = \|g^*|L_{p'}(\Gamma)\|. \tag{5.5.5}$$

This will be applied in the next proposition.

Proposition 5.5.5. *Let Γ be an h-set in $\mathbb{R}^n$, $1 \le p < \infty$, $0 < q < \infty$ and σ be an admissible sequence such that*

$$\underline{s}(\sigma) > \frac{1}{p}(n + \overline{s}(\boldsymbol{h})). \tag{5.5.6}$$

Then

$$\mathrm{tr}'_\Gamma = \mathrm{id}^\Gamma, \tag{5.5.7}$$

where id^Γ *is considered acting in* $L_{p'}(\Gamma)$*,* tr'_Γ *denotes the dual operator of*

$$\mathrm{tr}_\Gamma : B^\sigma_{p,q}(\mathbb{R}^n) \to L_p(\Gamma) \tag{5.5.8}$$

and where by (5.5.7) *we mean*

$$< \mathrm{tr}'_\Gamma\, g, v >=< \mathrm{id}^\Gamma\, g^*, v >, \quad v \in B^\sigma_{p,q}(\mathbb{R}^n), \tag{5.5.9}$$

for $g \in (L_p(\Gamma))'$ *and* $g^* \in L_{p'}(\Gamma)$ *according to* (5.5.4)-(5.5.5)*,*

Proof. It follows from (5.5.6), (2.1.3) and Proposition 3.2.2 that (5.5.8) makes sense. Therefore for the dual operator, tr'_Γ, we have

$$\mathrm{tr}'_\Gamma : (L_p(\Gamma))' \to (B^\sigma_{p,q}(\mathbb{R}^n))'.$$

By the definition of dual operator and (5.5.4), for all $v \in B^\sigma_{p,q}(\mathbb{R}^n)$,

$$< \mathrm{tr}'_\Gamma\, g, v >=< g, \mathrm{tr}_\Gamma\, v >= \int_\Gamma (\mathrm{tr}_\Gamma\, v)(\gamma) g^*(\gamma) d\mu(\gamma).$$

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$< \mathrm{tr}'_\Gamma\, g, \varphi >= \int_\Gamma (\varphi_{|\Gamma})(\gamma) g^*(\gamma) d\mu(\gamma) =< \mathrm{id}^\Gamma\, g^*, \varphi > .$$

Let $v \in B^{\sigma}_{p,q}(\mathbb{R}^n)$ and $(\varphi_j)_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ be such that

$$\|v - \varphi_j | B^{\sigma}_{p,q}(\mathbb{R}^n)\| \to 0 \quad \text{when} \quad j \to \infty.$$

Then

$$| < \operatorname{tr}'_\Gamma g, v - \varphi_j > | = | < g, \operatorname{tr}_\Gamma v - \varphi_{j|\Gamma} > | \leq \|g|(L_p(\Gamma))'\| \cdot \| \operatorname{tr}_\Gamma v - \varphi_{j|\Gamma}|L_p(\Gamma)\| \quad (5.5.10)$$

As $\|g|(L_p(\Gamma))'\|$ is finite, it follows from the definition of trace that the expression in (5.5.10) converges to 0 when $j \to \infty$. Hence

$$< \operatorname{tr}'_\Gamma g, v >= \lim_{j\to\infty} < \operatorname{tr}'_\Gamma g, \varphi_j >= \lim_{j\to\infty} < \operatorname{id}^\Gamma g^*, \varphi_j >=< \operatorname{id}^\Gamma g^*, v > . \quad (5.5.11)$$

We justify the last equality in (5.5.11): as $g^* \in L_{p'}(\Gamma)$, then, by Proposition 5.5.3 and Theorem 2.1.4,

$$\operatorname{id}^\Gamma g^* \in B^{\boldsymbol{h}^{-\frac{1}{p}}(n)^{-\frac{1}{p}}}_{p',\infty}(\mathbb{R}^n) = (B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,1}(\mathbb{R}^n))' \subset (B^{\sigma}_{p,q}(\mathbb{R}^n))'.$$

Hence

$$| < \operatorname{id}^\Gamma g^*, v - \varphi_j > | \leq \| \operatorname{id}^\Gamma g^* |(B^{\sigma}_{p,q}(\mathbb{R}^n))'\| \cdot \|v - \varphi_j|B^{\sigma}_{p,q}(\mathbb{R}^n)\| \to 0 \quad \text{when} \quad j \to \infty,$$

concluding the proof. □

Remark 5.5.6. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set satisfying the ball condition. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ such that $\Gamma \subset \Omega$. Consider $0 < p, q \leq \infty$ and an admissible sequence σ such that $\underline{s}(\sigma) > 0$. Let $f \in B^{\sigma}_{p,q}(\Omega)$ and $g, u \in B^{\sigma}_{p,q}(\mathbb{R}^n)$ be such that $g_{|\Omega} = u_{|\Omega} = f$. If for all such g and u one has $\operatorname{tr}_\Gamma g = \operatorname{tr}_\Gamma u$, then one can consider $\operatorname{tr}_\Gamma f := \operatorname{tr}_\Gamma g$, i.e., we can consider the operator trace acting in Besov spaces on Ω. Let $(\varphi_j)_j, (\psi_j)_j \subset \mathcal{S}(\mathbb{R}^n)$ converge in $B^{\sigma}_{p,q}(\mathbb{R}^n)$ to g and u, respectively. Let χ be a smooth function such that $\chi(x) = 1$ in a neighborhood of Γ and $\operatorname{supp}\chi \subset \Omega$. By Theorem 5.2.1, χg and χu are also elements of $B^{\sigma}_{p,q}(\mathbb{R}^n)$ and the sequences $(\chi\varphi_j)_j$ and $(\chi\psi_j)_j$ converge in $B^{\sigma}_{p,q}(\mathbb{R}^n)$ to χg and χu, respectively. By Definition 3.2.1,*

$$tr_\Gamma(\chi g) = \operatorname{tr}_\Gamma g \quad \text{and} \quad tr_\Gamma(\chi u) = \operatorname{tr}_\Gamma u.$$

So one just needs to prove that $\chi g = \chi u$*. Let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$*. Then*

$$< \chi g, \varphi >=< g, \chi\varphi >=< f, \chi\varphi >=< u, \chi\varphi >=< \chi u, \varphi > .$$

So

$$\mathrm{tr}_\Gamma\, B^\sigma_{p,q}(\Omega) := \mathrm{tr}_\Gamma\, B^\sigma_{p,q}(\mathbb{R}^n)$$

and all the results stated for the trace operator acting in Besov spaces on $\mathbb{R}^n$ *remain valid if one replaces* $\mathbb{R}^n$ *by* Ω*. Analogously for the results on the operators* id^Γ *and* tr^Γ*.*

5.6 The operator B

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and Γ be an h-set such that $\Gamma \subset \Omega$. We define

$$B := (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma, \tag{5.6.1}$$

where $(-\Delta)^{-1}$ denotes the inverse of the Dirichlet Laplacian presented in the previous section. In this section we study the operator B acting in Besov spaces of generalised smoothness on Ω.

By Proposition 5.5.4, Remark 5.5.6 and Theorem 5.4.4, if

$$1 \le p \le \infty \quad \text{and} \quad n - 2 < -\overline{s}(\boldsymbol{h})$$

then

$$B : B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,1}(\Omega) \to B^{(2)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}_{p,\infty}(\Omega).$$

The following proposition extends part of the results in [Mou01a, p. 123, Lemma 4.1.4].

Proposition 5.6.1. *Let* $h \in \mathbb{H}$ *be a strictly increasing function. Let* Ω *be a bounded* C^∞ *domain in* $\mathbb{R}^n$ *and* Γ *be an* h*-set such that* $\Gamma \subset \Omega$ *and*

$$(n-2)_+ < -\overline{s}(\boldsymbol{h}) \le -\underline{s}(\boldsymbol{h}) \le n. \tag{5.6.2}$$

Let

$$1 < p \le \infty, \qquad 0 < q \le \infty$$

and σ be an admissible sequence such that

$$\frac{n+\overline{s}(\boldsymbol{h})}{p} < \underline{s}(\sigma) \leq \overline{s}(\sigma) < 2 - \frac{n+\overline{s}(\boldsymbol{h})}{p'}. \tag{5.6.3}$$

Then, the operator

$$B := (-\Delta)^{-1} \circ \mathrm{tr}^{\Gamma}$$

is compact in $B^{\sigma}_{p,q}(\Omega)$. Moreover, if u is an eigenfunction associated to an eigenvalue of B, $\rho \neq 0$, then

$$u \in B^{(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega). \tag{5.6.4}$$

Remark 5.6.2. *As we are assuming that $-\overline{s}(\boldsymbol{h}) > n-2$, then*

$$2 - \frac{n+\overline{s}(\boldsymbol{h})}{p'} = 2 - (n+\overline{s}(\boldsymbol{h})) + \frac{n+\overline{s}(\boldsymbol{h})}{p} > \frac{n+\overline{s}(\boldsymbol{h})}{p}$$

and so condition (5.6.3) *makes sense.*

Proof. We prove that B is a compact operator acting in $B^{\sigma}_{p,q}(\Omega)$. We factorize B as follows

$$B = \mathrm{id}_2 \circ (-\Delta)^{-1} \circ \mathrm{id}^{\Gamma} \circ \mathrm{tr}_{\Gamma} \circ \mathrm{id}_1,$$

where, for a number $s \in ((n+\overline{s}(\boldsymbol{h}))/p, \underline{s}(\sigma))$,

$$\mathrm{id}_1 \;:\; B^{\sigma}_{p,q}(\Omega) \hookrightarrow B^{(s)}_{p,p}(\Omega) \tag{5.6.5}$$

$$\mathrm{tr}_{\Gamma} \;:\; B^{(s)}_{p,p}(\Omega) \to L_p(\Gamma) \tag{5.6.6}$$

$$\mathrm{id}^{\Gamma} \;:\; L_p(\Gamma) \to B^{\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega) \tag{5.6.7}$$

$$(-\Delta)^{-1} \;:\; B^{\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega) \to B^{(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega) \tag{5.6.8}$$

$$\mathrm{id}_2 \;:\; B^{(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega) \hookrightarrow B^{\sigma}_{p,q}(\Omega). \tag{5.6.9}$$

The embedding (5.6.5) can be justified by Proposition 2.1.5 and the property in (1.2.4), because $s < \underline{s}(\sigma)$.

By [Tri04, p. 13, Theorem 1], the operator tr_{Γ} in (5.6.6) is compact if

$$\sum_{j=0}^{\infty} 2^{-jp'(s-\frac{n}{p})} h(2^{-j})^{p'-1} < \infty. \tag{5.6.10}$$

Let $\delta \in (0, sp - (n + \overline{s}(\boldsymbol{h})))$. By the property in (1.2.4), there is a positive number c such that, for all $j \in \mathbb{N}_0$,

$$2^{-jp'(s-\frac{n}{p})} h(2^{-j})^{p'-1} \leq c 2^{-jp'(s-\frac{n+\overline{s}(\boldsymbol{h})}{p}-\frac{\delta}{p})}$$

and so (5.6.10) is satisfied and tr_Γ is compact. The embeddings (5.6.7) and (5.6.8) are justified by Proposition 5.5.3 and Theorem 5.4.4, respectively.

Finally, (5.6.9) follows from Proposition 2.1.5 and Remark 5.0.6 making use of (5.6.3). From the previous factorization of B, (5.6.4) follows immediately. □

Remark 5.6.3. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and Γ be an h-set such that $\Gamma \subset \Omega$ and*

$$(n-2)_+ < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) \leq n.$$

Then taking $p = q = 2$ and $\sigma = (1)$, conditions of Proposition 5.6.1 are satisfied. Hence, B is a compact operator in $H^1(\Omega)$. Now let $1 < p \leq \infty$. Although the sequence $\sigma = (2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}$ does not necessarily satisfy (5.6.3), one can conclude, from the proof of Proposition 5.6.1, that the operator B is compact in $B_{p,\infty}^{(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}(\Omega)$, considering the factorization

$$B = \mathrm{id}_2 \circ (-\Delta)^{-1} \circ \mathrm{id}^\Gamma \circ \mathrm{tr}_\Gamma \circ \mathrm{id}_1 \circ \mathrm{id}_2,$$

where the operators involved are as in (5.6.5)-(5.6.9).

5.7 The operator B in $\overset{\circ}{H}^1(\Omega)$

In this section we consider the operator B acting in $\overset{\circ}{H}^1(\Omega)$. We refer to Definition 5.0.4(ii) and Remarks 2.1.3 and 5.0.5. We prove that B acting in this space is self-adjoint, non-negative, compact and generated by a sesquilinear form. We deal with the asymptotic distribution of the eigenvalues of B and present some results on the associated eigenfunctions.

Definition 5.7.1. *Let Ω be a domain in $\mathbb{R}^n$. We define, for $f, g \in \mathring{H}^1(\Omega)$,*

$$(f,g)_{\mathring{H}^1(\Omega)} := \sum_{j=1}^{n} \int_\Omega \frac{\partial f}{\partial x_j}(x) \frac{\partial \overline{g}}{\partial x_j}(x) dx, \tag{5.7.1}$$

where $\frac{\partial f}{\partial x_j}$ and $\frac{\partial f}{\partial x_j}$ denote, for $j \in \{1, ..., n\}$, the weak derivatives of first order of f and $\overline{g}$, respectively.

The following result follows from Friedrich's inequality (cf. [Tri92a, p. 357], for example) and it can be found in [Tri97, p. 195].

Proposition 5.7.2. *Let Ω be a bounded domain in $\mathbb{R}^n$. For all $f \in \mathring{H}^1(\Omega)$*

$$\|f|H^1(\Omega)\|^2 \sim (f,f)_{\mathring{H}^1(\Omega)}.$$

Remark 5.7.3. *In what follows, for technical reasons, in the space $\mathring{H}^1(\Omega)$ we will not consider the norm inherited from $H^1(\Omega)$, but the equivalent norm given by*

$$\|f|\mathring{H}^1(\Omega)\| := \sqrt{(f,f)_{\mathring{H}^1(\Omega)}} = \sqrt{\sum_{j=1}^{n} \int_\Omega \left|\frac{\partial f}{\partial x_j}(x)\right|^2 dx}, \quad f \in \mathring{H}^1(\Omega), \tag{5.7.2}$$

which can be done according to Proposition 5.7.2.

In the proof of the following theorem we will apply the notion of approximation numbers of an operator. We recall the definition.

Definition 5.7.4. *Let X and Y be normed vector spaces. Given $L \in L(X,Y)$, rank L denotes the dimension of the range of L. Let $T \in L(X,Y)$ and $k \in \mathbb{N}$. The kth approximation number, $a_k(T)$, of T is defined by*

$$a_k(T) := \inf\{\|T - L\| : L \in L(X,Y) \text{ and rank} \, L < k\}.$$

In the following proof we will also apply the next result.

Proposition 5.7.5. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $n \geq 2$. There is a function G,*

$$G : \overline{\Omega} \times \Omega \to \mathbb{R},$$

usually called Green's function such that *(i) for all* $x^0 \in \Omega$ *and* $\varepsilon > 0$,

$$G(x^0, \cdot) \in C^\infty(\Omega \setminus \overline{B(x^0, \varepsilon)});$$

(ii) for all $x, y \in \Omega$, *with* $x \neq y$, $G(x, y) = G(y, x)$;

(iii) if $n \geq 3$

$$0 < G(x, y) \lesssim |x - y|^{2-n}, \quad x, y \in \Omega, \quad x \neq y,$$

and, if $n = 2$,

$$0 < G(x, y) \lesssim \max_{z \in \partial\Omega} \ln|z - x| - \ln|x - y|, \quad x, y \in \Omega, \quad x \neq y;$$

(iv) $G(x, y) = 0$, *for all* $x \in \partial\Omega$ *and* $y \in \Omega$;

(v) for all $\varphi \in \mathcal{D}(\Omega)$,

$$(-\Delta)^{-1}\varphi(x) = \int_\Omega G(x, y)\varphi(y)dy, \quad x \in \Omega. \tag{5.7.3}$$

Therefore, for all $f \in H^{-1}(\Omega)$,

$$(-\Delta)^{-1} f = \lim_{j \to \infty} \int_\Omega G(\cdot, y)\varphi_j(y)dy \quad \text{in} \quad \mathring{H}^1(\Omega),$$

where

$$(\varphi_j)_{j \in \mathbb{N}_0} \subset \mathcal{D}(\Omega) \quad \text{with} \quad f = \lim_{j \to \infty} \varphi_j \quad \text{in} \quad H^{-1}(\Omega). \tag{5.7.4}$$

Remark 5.7.6. *In* (5.7.4) *we used the fact that* $\mathcal{D}(\Omega)$ *is dense in* $H^{-1}(\Omega)$, *which can be justified as follows: the restriction of* $\mathcal{S}(\mathbb{R}^n)$ *to* Ω *is dense in* $H^{-1}(\Omega)$. *Any function in the restriction of* $\mathcal{S}(\mathbb{R}^n)$ *to* Ω *can be approximated in* $L_2(\Omega)$ *by functions belonging to* $\mathcal{D}(\Omega)$. *But this is also an approximation in* $H^{-1}(\Omega)$.
For the results in the previous proposition we refer to [Tri01, p. 299], [HT08, pp. 160-163, 273].
In connection with the Green's function we also refer to [Tri92a, pp. 145, 194-196], [Tri01, pp. 243-244], [Tri06, p. 301] and [AH96, pp. 10-13].

The next theorem extends the results in [Tri01, Theorem 19.7] and [ET99, Theorem 2.28]. We also refer to [Mou01a, Theorem 4.1.7].

Theorem 5.7.7. *Let $h \in \mathbb{H}$ be a strictly increasing function. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $n \geq 2$, and Γ be an h-set such that $\Gamma \subset \Omega$ and*

$$n - 2 < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) < n. \tag{5.7.5}$$

Then B is a non-negative compact self-adjoint operator in $\mathring{H}^1(\Omega)$ with null-space

$$N(B) = \{f \in \mathring{H}^1(\Omega) : \mathrm{tr}_\Gamma\, f = 0\}. \tag{5.7.6}$$

Moreover, B is generated by the sesquilinear form

$$(Bf, g)_{\mathring{H}^1(\Omega)} = \int_\Gamma (\mathrm{tr}_\Gamma\, f)(\gamma)\overline{(\mathrm{tr}_\Gamma\, g)(\gamma)}d\mu(\gamma), \quad f, g \in \mathring{H}^1(\Omega), \tag{5.7.7}$$

with (5.7.1) as the scalar product in $\mathring{H}^1(\Omega)$. Furthermore, B is given by

$$Bf = \int_\Gamma G(\cdot, \gamma)(\mathrm{tr}_\Gamma\, f)(\gamma)d\mu(\gamma), \quad f \in \mathring{H}^1(\Omega), \tag{5.7.8}$$

where G is the Green's function referred in Proposition 5.7.5.
Let ρ_k denote the positive eigenvalues of B repeated according to multiplicity and ordered by decreasing order of their magnitude and u_k denote related eigenfunctions

$$Bu_k = \rho_k u_k, \quad k \in \mathbb{N}.$$

(i) The largest eigenvalue is simple, i.e.,

$$\rho_1 > \rho_2 \geq \rho_3 \ldots$$

and, for all $k \in \mathbb{N}$,

$$\rho_k \sim k^{-1} H(k^{-1})^{2-n}, \tag{5.7.9}$$

where H denotes the inverse function of h.

(ii) The eigenfunctions u_k are (classical) harmonic functions in $\Omega \setminus \Gamma$,

$$\Delta u_k(x) = 0 \quad \text{if} \quad x \in \Omega \setminus \Gamma. \tag{5.7.10}$$

(iii) Let $1 < p \leq \infty$, $\varepsilon \in \mathbb{R}$ and p' be the conjugate exponent of p. Then

$$u_k \in B^{(\varepsilon)(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega) \quad \text{if, and only if,} \quad \varepsilon \leq 0.$$

(iv) The eigenfunctions $u_1(x)$ have no zeros in Ω

$$u_1(x) = cu(x) \quad \text{with} \quad c \in \mathbb{C} \quad \text{and} \quad u(x) > 0 \quad \text{if} \quad x \in \Omega.$$

Proof. Step 1: Let us prove that B is a compact operator acting in $\mathring{H}^1(\Omega)$. We factorize B as

$$B = (-\Delta)^{-1} \circ \mathrm{id}_3 \circ \mathrm{id}^\Gamma \circ \mathrm{tr}_\Gamma \,.$$

where

$$\mathrm{tr}_\Gamma \quad : \quad \mathring{H}^1(\Omega) \to L_2(\Gamma) \tag{5.7.11}$$

$$\mathrm{id}^\Gamma \quad : \quad L_2(\Gamma) \to B^{\boldsymbol{h}^{-1/2}(n)^{-1/2}}_{2,\infty}(\Omega) \tag{5.7.12}$$

$$\mathrm{id}_3 \quad : \quad B^{\boldsymbol{h}^{-1/2}(n)^{-1/2}}_{2,\infty}(\Omega) \hookrightarrow H^{-1}(\Omega) \tag{5.7.13}$$

$$(-\Delta)^{-1} \quad : \quad H^{-1}(\Omega) \to \mathring{H}^1(\Omega) \tag{5.7.14}$$

In the conditions considered (5.6.10) is satisfied for $s = 1$ and $p = p' = 2$, so, by [Tri04, p. 13, Theorem 1], tr_Γ in (5.7.11) is compact. Proposition 5.5.3 justifies (5.7.12) and the embedding (5.7.13) follows from (2.1.3), because

$$\underline{s}(\boldsymbol{h}^{-1/2}(n)^{-1/2}) = -\frac{\overline{s}(\boldsymbol{h})}{2} - \frac{n}{2} > \frac{n-2}{2} - \frac{n}{2} = -1 = \overline{s}((-1)).$$

By [Tri01, p. 255],

$$\mathring{H}^1(\Omega) = \{f \in H^1(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\}.$$

Therefore, applying Theorem 5.4.1, we obtain (5.7.14).

Step 2: Let us prove (5.7.8). Let $f \in \mathring{H}^1(\Omega)$. Then, by (5.7.11)-(5.7.13), $\mathrm{id}^\Gamma(\mathrm{tr}_\Gamma f)$ belongs to $H^{-1}(\Omega)$. Let $(\psi_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ converge to $\mathrm{id}^\Gamma(\mathrm{tr}_\Gamma f)$ in $H^{-1}(\Omega)$. Let us consider $\chi_k \in \mathcal{D}(\Omega)$, $k \in \mathbb{N}$, such that

$$0 \leq \chi_k \leq 1, \quad \chi_k(x) = 1, \text{ if } x \in \Gamma, \quad \text{and} \quad \operatorname{supp} \chi_k \subset \Gamma_{1/k}. \tag{5.7.15}$$

Applying Corollary 5.2.2 one concludes that, for all $k \in \mathbb{N}$,

$$\lim_{j\to\infty} (\chi_k \psi_j) = \chi_k \,\mathrm{id}^\Gamma(\mathrm{tr}_\Gamma f) \quad \text{in} \quad H^{-1}(\Omega).$$

By Definition 5.5.1 and (5.7.15)

$$\chi_k \,\mathrm{id}^\Gamma(\mathrm{tr}_\Gamma f) = \mathrm{id}^\Gamma(\mathrm{tr}_\Gamma f), \quad k \in \mathbb{N}.$$

By Proposition 5.7.5, for each $k \in \mathbb{N}$,

$$Bf = (-\Delta)^{-1} \mathrm{id}^\Gamma(\mathrm{tr}_\Gamma f) = \lim_{j\to\infty} \int_\Omega G(\cdot, y)\chi_k(y)\psi_j(y)dy \quad \text{in} \quad \mathring{H}^1(\Omega).$$

So, for each $k \in \mathbb{N}$, there is $A_k \subset \mathbb{R}^n$ with $|A_k| = 0$ and there is a subsequence of $(\chi_k\psi_j)_{j\in\mathbb{N}}$, say $(\chi_k\psi_{\sigma_k(j)})_{j\in\mathbb{N}}$, such that

$$Bf(x) = \lim_{j\to\infty} \int_\Omega G(x, y)\chi_k(y)\psi_{\sigma_k(j)}(y)dy, \quad \text{for all} \quad x \in \Omega \setminus A_k. \tag{5.7.16}$$

Let

$$A := \bigcup_{k=1}^{\infty} A.$$

Then $|A| = 0$ and (5.7.16) is satisfied for all $k \in \mathbb{N}$ and all $x \in \Omega \setminus A$. But we remark that for each k we may have a different subsequence $(\chi_k\psi_{\sigma_k(j)})_j$. As we want to obtain a representation for Bf, which is a regular distribution, we may exclude from consideration sets with Lebesgue measure equal to 0. So it is enough to consider $x \in \Omega \setminus (A \cup \Gamma)$. Let us consider such x. We fix $k \in \mathbb{N}$ such that $2/k < \min\{\operatorname{dist}(x, \Gamma), \operatorname{dist}(\Gamma, \partial\Omega)\}$ and we consider $\theta_k \in \mathcal{D}(\Omega)$ satisfying

$$0 \leq \theta_k \leq 1, \quad \theta_k(y) = 1, \text{ if } y \in \Gamma_{1/k}, \quad \text{and} \quad \operatorname{supp} \theta_k \subset \Gamma_{2/k}.$$

Hence

$$\begin{aligned} Bf(x) &= \lim_{j\to\infty}\int_\Omega G(x,y)\chi_k(y)\psi_{\sigma_k(j)}(y)dy \\ &= \lim_{j\to\infty}\int_\Omega G(x,y)\theta_k(y)\chi_k(y)\psi_{\sigma_k(j)}(y)dy. \end{aligned}$$

We remark that $G(x,\cdot)\theta_k \in \mathcal{D}(\Omega)$ and that $(\chi_k\psi_{\sigma_k(j)})_j$ converges to $\mathrm{id}^\Gamma(\mathrm{tr}_\Gamma f)$ in $H^{-1}(\Omega)$ and, so, also in $\mathcal{D}'(\Omega)$. Thus

$$\begin{aligned} Bf(x) &= \lim_{j\to\infty} < \chi_k\psi_{\sigma_k(j)}, G(x,\cdot)\theta_k > \\ &= < \mathrm{id}^\Gamma(\mathrm{tr}_\Gamma f), G(x,\cdot)\theta_k > \\ &= \int_\Gamma (\mathrm{tr}_\Gamma f)(\gamma)G(x,\gamma)\theta_k(\gamma)d\mu(\gamma) \\ &= \int_\Gamma (\mathrm{tr}_\Gamma f)(\gamma)G(x,\gamma)d\mu(\gamma), \end{aligned}$$

proving (5.7.8).

Step 3: Let us prove (5.7.7). Let $f \in \mathring{H}^1(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Then

$$\begin{aligned} (Bf,\varphi)_{\mathring{H}^1(\Omega)} &= \sum_{j=1}^n \Big\langle \frac{\partial Bf}{\partial x_j}, \frac{\partial\overline{\varphi}}{\partial x_j}\Big\rangle \\ &= -\sum_{j=1}^n \Big\langle \frac{\partial^2 Bf}{\partial x_j^2}, \overline{\varphi}\Big\rangle \\ &= \big\langle -\Delta Bf, \overline{\varphi}\big\rangle \\ &= \big\langle -\Delta\circ(-\Delta)^{-1}\circ \mathrm{tr}^\Gamma f, \overline{\varphi}\big\rangle \\ &= \big\langle \mathrm{tr}^\Gamma f, \overline{\varphi}\big\rangle \\ &= \int_\Gamma (\mathrm{tr}_\Gamma f)(\gamma)\overline{(\varphi|_\Gamma(\gamma))}d\mu(\gamma). \end{aligned}$$

Let $f, g \in \mathring{H}^1(\Omega)$ and $(\varphi_t)_{t\in\mathbb{N}_0} \subset \mathcal{D}(\Omega)$ convergent to g in $\mathring{H}^1(\Omega)$. Then

$$\begin{aligned} \Big|(Bf,g)_{\mathring{H}^1(\Omega)} - \int_\Gamma (\mathrm{tr}_\Gamma f)(\gamma)\overline{(\mathrm{tr}_\Gamma g)(\gamma)}d\mu(\gamma)\Big| &\le \lim_{t\to\infty}\int_\Gamma |(\mathrm{tr}_\Gamma f)(\gamma)\overline{(\mathrm{tr}_\Gamma \varphi_t - \mathrm{tr}_\Gamma g)(\gamma)}|d\mu(\gamma) \\ &\le \lim_{t\to\infty} \|\mathrm{tr}_\Gamma f|L_2(\Gamma)\|\cdot\|\mathrm{tr}_\Gamma(\varphi_t - g)|L_2(\Gamma)\| \\ &\le c\|\mathrm{tr}_\Gamma f|L_2(\Gamma)\| \lim_{t\to\infty}\|\varphi_t - g|H^1(\Omega)\| \\ &= 0 \end{aligned}$$

Hence, we proved (5.7.7) and we also conclude (5.7.6) and that B is a non-negative self-adjoint operator in $\mathring{H}^1(\Omega)$.

Step 4: We prove (5.7.9). We can easily check that conditions $n \geq 2$ and (5.7.5) guarantee that [Tri04, p. 14, Theorem 2] can be applied to conclude that the approximation numbers of the compact operator

$$\mathrm{tr}_\Gamma : \mathring{H}^1(\Omega) \to L_2(\Gamma)$$

satisfy

$$a_k(\mathrm{tr}_\Gamma) \sim k^{-\frac{1}{2}} H(k^{-1})^{1-\frac{n}{2}}, \quad k \in \mathbb{N}.$$

By (5.5.7) and applying the known assertion about the relation between the approximation numbers of dual operators (cf. [EE87, p. 55, Proposition II, 2.5], for example),

$$a_k(\mathrm{id}^\Gamma) = a_k(\mathrm{tr}_\Gamma) \sim k^{-\frac{1}{2}} H(k^{-1})^{1-\frac{n}{2}}, \quad k \in \mathbb{N}.$$

Hence, again by the properties of the approximation numbers (cf. [EE87, p. 53, Proposition II, 2.2]),

$$a_{2k}(B) \leq c\, a_k(\mathrm{tr}_\Gamma) a_k(\mathrm{id}^\Gamma) \sim k^{-1} H(k^{-1})^{2-n}, \quad k \in \mathbb{N}.$$

By the properties of h and as $a_k(B) = \rho_k$ (cf. [EE87, p 91, Theorem II, 5.10]) we get to the "$\lesssim$" part of (5.7.9).

Let us prove the "$\gtrsim$" part of (5.7.9). Let ρ_k be a positive eigenvalue of B. Then $\sqrt{\rho}_k$ is an eigenvalue for the operator $\sqrt{B}$. So again by [EE87, p 91, Theorem II, 5.10] it is sufficient to prove that

$$a_k(\sqrt{B}) \gtrsim k^{-\frac{1}{2}} H(k^{-1})^{1-\frac{n}{2}}, \quad k \in \mathbb{N}. \tag{5.7.17}$$

As $\Gamma \subset \Omega$, $\mathrm{dist}(\Gamma, \partial\Omega) > 0$. Let $j_0 \in \mathbb{N}$ be such that $2^{-j_0} < \mathrm{dist}(\Gamma, \partial\Omega)$. For all $j \in \mathbb{N}$, $j \geq j_0$, there is $\{\gamma^{j,m}\}_{m=1}^{M_j} \subset \Gamma$ such that

$$B(\gamma^{j,m_1}, 2^{-j}) \cap B(\gamma^{j,m_2}, 2^{-j}) = \emptyset, \quad \text{if} \quad m_1 \neq m_2, \tag{5.7.18}$$

$$B(\gamma^{j,m}, 2^{-j}) \subset \Omega, \quad l = 1, ..., M_j,$$

and

$$M_j \sim \frac{1}{h(2^{-j})}. \tag{5.7.19}$$

The equivalence (5.7.19) follows from [Bri01, p. 30, Lemma 1.8.3]. Let $\delta \in (0, 1/2)$ and ω be a C^∞ non-negative function such that

$$\operatorname{supp} \omega \subset \{x \in \mathbb{R}^n : |x| \leq 1/2\}$$

and

$$\omega(x) > 0, \quad \text{if} \quad |x| \leq \delta.$$

Let

$$\omega^{j,m}(x) := \omega(2^j(x - \gamma^{j,m})), \quad j \geq j_0, \quad m = 1, ..., M_j,$$

and

$$f_j(x) := \sum_{m=1}^{M_j} c_{j,m}\omega^{j,m}(x), \quad c_{j,m} \in \mathbb{C}, \quad j \geq j_0. \tag{5.7.20}$$

Then, as $\operatorname{supp} \omega^{j,m} \subset B(\gamma^{j,m}, 2^{-j})$, by (5.7.2) and (5.7.18),

$$\|f_j|\mathring{H}^1(\Omega)\|^2 = \sum_{m=1}^{M_j} |c_{j,m}|^2 \cdot \|\omega^{j,m}|\mathring{H}^1(\Omega)\|^2.$$

By the conditions on the support of ω and applying the homogeneity property (cf. [CLT07]),

$$\|\omega^{j,m}|\mathring{H}^1(\Omega)\| \sim \|\omega^{j,m}|H^1(\mathbb{R}^n)\| \sim 2^{j(1-\frac{n}{2})}\|\omega|H^1(\mathbb{R}^n)\|$$

Therefore

$$\|f_j|\mathring{H}^1(\Omega)\| \sim 2^{j(1-\frac{n}{2})}\Big(\sum_{m=1}^{M_j} |c_{j,m}|^2\Big)^{1/2}. \tag{5.7.21}$$

For all $j \geq j_0$, applying also (5.7.18),

$$\begin{aligned}
\| \operatorname{tr}_\Gamma f_j | L_2(\Gamma)\|^2 &= \sum_{m=1}^{M_j} |c_{j,m}|^2 \int_\Gamma (\omega^{j,m}(\gamma))^2 d\mu(\gamma) \\
&\geq \sum_{m=1}^{M_j} |c_{j,m}|^2 \int_{\Gamma \cap B(\gamma^{j,m}, \delta 2^{-j})} (\omega^{j,m}(\gamma))^2 d\mu(\gamma) \\
&\geq \inf_{|x| \leq \delta} \omega(x)^2 \cdot \sum_{m=1}^{M_j} |c_{j,m}|^2 \cdot \mu(B(\gamma^{j,m}, \delta 2^{-j})) \\
&\sim h(2^{-j}) \sum_{m=1}^{M_j} |c_{j,m}|^2
\end{aligned}$$

Analogously one can obtain the reverse inequality. So

$$\| \operatorname{tr}_\Gamma f_j | L_2(\Gamma)\| \sim h(2^{-j})^{1/2} \Big(\sum_{m=1}^{M_j} |c_{j,m}|^2 \Big)^{1/2}. \tag{5.7.22}$$

Let $T \in L(\mathring{H}^1(\Omega), \mathring{H}^1(\Omega))$ be such that rank $T < M_j$. Then there is $(c_{j,m})_{m=1}^{M_j} \subset \mathbb{C}$ such that

$$\sum_{m=1}^{M_j} c_{j,m} T \omega^{j,m} = 0, \tag{5.7.23}$$

where there is at least one $m = 1, ..., M_j$ for which $c_{j,m} \neq 0$. Let f_j be given by (5.7.20). By (5.7.21)-(5.7.23),

$$\|\sqrt{B} - T\| \geq \frac{\|\sqrt{B} f_j | \mathring{H}^1(\Omega)\|}{\|f_j | \mathring{H}^1(\Omega)\|} \sim h(2^{-j})^{1/2} 2^{-j(1-\frac{n}{2})}. \tag{5.7.24}$$

Now (5.7.17) follows from (5.7.19), (5.7.24) and the properties of h.

Step 5: We prove *(ii)*. Let ρ_k be a positive eingenvalue for the operator B and u_k an associated eigenfunction. Hence

$$\rho_k^{-1} \operatorname{tr}^\Gamma u_k = (-\Delta) u_k \in B_{2,\infty}^{\boldsymbol{h}^{-1/2}(n)^{-1/2}}(\Omega).$$

As $\operatorname{supp}(\operatorname{tr}^\Gamma u_k) \subset \Gamma$ then $\Delta u_k = 0$ in $\mathcal{D}'(\Omega \setminus \Gamma)$. It is a well-known result of the theory of distributions that a distributional solution of the Laplace equation is also a classic solution (cf. [Tri95, Section 6.4.1, Lemma, p. 414]), so we conclude (5.7.10).

Step 6: We prove *(iii)*. Let us prove the "if" part. Let

$$1 < p_0 < p_1 \leq \infty \quad \text{and} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$$

and assume that (5.6.3) is satisfied considering $\sigma = (s_i)$ and $p = p_i$, for $i = 0, 1$. Then

$$B^{(s_0)}_{p_0,q}(\Omega) \subset B^{(s_1)}_{p_1,q}(\Omega) \subset B^{(s_1)}_{p_0,q}(\Omega). \tag{5.7.25}$$

For the first inclusion in (5.7.25) we refer to [Tri83, 2.7.1, p. 129] and for the second one we refer to [Mou01a, p. 120, Proposition 4.1.2]. Let us temporarily denote by $B(p_i, s_j)$, $i, j = 0, 1$, the operator B considered acting in $B^{(s_j)}_{p_i,q}(\Omega)$. By (5.7.25), the eigenvalues and the related eigenfunctions of $B(p_0, s_0)$ are also eigenvalues and related eigenfunctions for $B(p_1, s_1)$ and $B(p_0, s_1)$. Now let $\rho \neq 0$ be an eigenvalue for $B(p_0, s_1)$ and u be an associated eigenfunction. Then, by Proposition 5.6.1, $u \in B^{(2)\boldsymbol{h}^{-1/p_0'}(n)^{-1/p_0'}}_{p_0,\infty}(\Omega)$. The embedding (5.6.9) with $\sigma = (s_0)$ and $p = p_0$ holds and this can be justified as in the proof of Proposition 5.6.1, because (5.6.3) is satisfied. So $u \in B^{(s_0)}_{p_0,q}(\Omega)$ and thus ρ and u are an eigenvalue and an associated eigenfunction, respectively, for $B(p_0, s_0)$ and $B(p_1, s_1)$. Therefore the eigenvalues $\rho \neq 0$ and the associated eigenfunctions of $B(p_0, s_0)$, $B(p_1, s_1)$ and $B(p_0, s_1)$ coincide.

Let $1 < p \leq \infty$. If $p = 2$ it is immediate by Proposition 5.6.1 that if u is an eigenfucntion for B then $u \in B^{(2)\boldsymbol{h}^{-1/2}(n)^{-1/2}}_{2,\infty}(\Omega)$. Otherwise, we consider a number s such that (5.6.3) is satisfied for $\sigma = (s)$. Using pairs as (p_0, s_0) and (p_1, s_1) and repeating this construction a finite number of times one can "reach" the pair (p, s) starting from $(2, 1)$, which corresponds to the space $B^{(1)}_{2,2}(\Omega) = H^1(\Omega)$. One can replace $H^1(\Omega)$ by $\mathring{H}^1(\Omega)$, which does not influence the above arguments. Therefore one concludes that the eigenfunctions of B belong to $B^{(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega)$.

Now let us prove the "only if" part. Assume that there are $\varepsilon > 0$, $1 < p \leq \infty$ and an eigenfunction u_k associated to a positive eigenvalue ρ_k of the operator B such that

$$u_k \in B^{(2+\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}_{p,\infty}(\Omega).$$

Let $\varphi \in \mathcal{D}(\Omega)$ with $\varphi_{|\Gamma} = 0$. Then

$$-\rho_k < \Delta u_k, \varphi > = < \operatorname{tr}^\Gamma u_k, \varphi > = < \operatorname{id}^\Gamma \circ \operatorname{tr}_\Gamma u_k, \varphi > = \int_\Gamma (\operatorname{tr}_\Gamma u_k)(\gamma)\varphi(\gamma)d\mu(\gamma) = 0$$

and so

$$\Delta u_k \in B^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}_{p,\infty}(\Omega)$$

and

$$< \Delta u_k, \psi >= 0, \quad \text{for all} \quad \psi \in \mathcal{D}(\Omega) \quad \text{such that} \quad \psi_{|\Gamma} = 0. \tag{5.7.26}$$

Let us consider two smooth positive functions η_1 and η_2 such that $\eta_1(x) + \eta_2(x) = 1$, for all $x \in \mathbb{R}^n$,

$$\operatorname{supp} \eta_1 \subset \Omega \quad \text{and} \quad \operatorname{supp} \eta_2 \subset (\mathbb{R}^n \setminus \Omega)_r, \tag{5.7.27}$$

where $(\mathbb{R}^n \setminus \Omega)_r$ denotes a neighborhood of $\mathbb{R}^n \setminus \Omega$ such that $\Gamma \cap (\mathbb{R}^n \setminus \Omega)_r = \emptyset$. We define

$$< \widetilde{\Delta u_k}, \varphi >:=< \Delta u_k, (\eta_1 \varphi)_{|\Omega} >, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{5.7.28}$$

Let us prove that

$$\widetilde{\Delta u_k} \in B^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}},\Gamma}_{p,\infty}(\mathbb{R}^n) \quad \text{and} \quad (\widetilde{\Delta u_k})_{|\Omega} = \Delta u_k, \tag{5.7.29}$$

where we are using the notation introduced in Definition 5.5.2. Let $\psi \in \mathcal{D}(\Omega)$ and $\widetilde{\psi}$ denote the extension of ψ to $\mathbb{R}^n$ by zero. Then

$$< (\widetilde{\Delta u_k})_{|\Omega}, \psi >=< \widetilde{\Delta u_k}, \widetilde{\psi} >=< \Delta u_k, (\eta_1 \widetilde{\psi})_{|\Omega} > .$$

By (5.7.26) and (5.7.27), one concludes that $< \Delta u_k, (\eta_2 \widetilde{\psi})_{|\Omega} >= 0$ and so

$$< (\widetilde{\Delta u_k})_{|\Omega}, \psi >=< \Delta u_k, (\eta_1 \widetilde{\psi} + \eta_2 \widetilde{\psi})_{|\Omega} >=< \Delta u_k, \psi >,$$

proving the second part of (5.7.29). Let us prove the first part. Let $g \in B^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}_{p,\infty}(\mathbb{R}^n)$ be such that $g_{|\Omega} = \Delta u_k$. By Theorem 5.2.1, $\eta_1 g \in B^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}_{p,\infty}(\mathbb{R}^n)$. By the definition of $\widetilde{\Delta u_k}$ one can easily conclude that $\widetilde{\Delta u_k} = \eta_1 g$ and so $\widetilde{\Delta u_k} \in B^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}_{p,\infty}(\mathbb{R}^n)$. Now let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\varphi_{|\Gamma} = 0$. By (5.7.26)-(5.7.28), one can conclude that $< \widetilde{\Delta u_k}, \varphi >= 0$ and so (5.7.29) is proved.

Let $\delta \in (0, \varepsilon p'/2)$. Let us apply (2.1.2) to conclude that

$$B^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}},\Gamma}_{p,\infty}(\mathbb{R}^n) \subset B^{\left(-\frac{n-(\omega(h)+\delta)}{p'}\right),\Gamma}_{p,\infty}(\mathbb{R}^n). \tag{5.7.30}$$

By Definition 3.1.6 there is $j_0 \in \mathbb{N}$ such that

$$\underline{\omega}(h) - \delta \leq \frac{\log h(2^{-j})}{\log 2^{-j}}, \quad j \geq j_0.$$

Then, for all $j \geq j_0$,

$$2^{-\varepsilon j} h(2^{-j})^{\frac{1}{p'}} 2^{\frac{nj}{p'}} 2^{-\frac{n-(\underline{\omega}(h)+\delta)}{p'} j} \leq 2^{-\varepsilon j} 2^{\frac{-j(\underline{\omega}(h)-\delta)}{p'}} 2^{\frac{nj}{p'}} 2^{-\frac{n-(\underline{\omega}(h)+\delta)}{p'} j} \leq 2^{-j(\varepsilon - \frac{2\delta}{p'})}. \tag{5.7.31}$$

Now (5.7.30) follows from (2.1.2) and (5.7.31). As it was mentioned in Proposition 3.1.7, the Hausdorff dimension of an h-set Γ, $\dim_{\mathcal{H}}\Gamma$, is equal to $\underline{\omega}(h)$. By [Tri97, p. 130, Theorem 17.8],

$$\dim_{\mathcal{H}}\Gamma = \sup\left\{t \; : \; B_{p,\infty}^{(-\frac{n-t}{p'}),\Lambda}(\mathbb{R}^n) \text{ is non-trivial for some compact } \Lambda \subset \Gamma\right\}.$$

Therefore both spaces in (5.7.30) are trivial. So $\widetilde{\Delta u_k} = 0$ and $\Delta u_k = 0$. By Theorem 5.4.4, $-\Delta$ is an isomorphism. Then $u_k = 0$, which is a contradiction.

Step 7: Let $\rho = \rho_1$ be the largest positive eigenvalue of B. Let us prove that a non-trivial function $v \in \mathring{H}^1(\Omega)$ is an eigenfunction of B associated to ρ if, and only if,

$$\int_\Gamma |(\mathrm{tr}_\Gamma\, v)(\gamma)|^2 d\mu(\gamma) = \rho \sum_{j=1}^n \int_\Omega \left|\frac{\partial v}{\partial x_j}(x)\right|^2 dx. \tag{5.7.32}$$

The "only if" part is immediate (cf. (5.7.7)). We prove the "if" one. First, we introduce some more notation: we denote by $(f_k)_{k\in\mathbb{N}} \subset \mathring{H}^1(\Omega)$ an orthonormal system of eigenfunctions associated to the corresponding eigenvalues ρ_k, i.e., $Bf_k = \rho_k f_k$, $k \in \mathbb{N}$, and by M the closed subspace generated by $(f_k)_k$. By Theorem 4.4 in [TL80, p. 357],

$$\mathring{H}^1(\Omega) = M \oplus N(B).$$

Let $v \in \mathring{H}^1(\Omega)$, $v \neq 0$, satisfy (5.7.32). So there exist $z \in M$ and $g \in N(B)$ such that

$$v = z + g. \tag{5.7.33}$$

As $z \in M$, there exist $(\alpha_k)_k \subset \mathbb{C}$ such that

$$z = \sum_{k=1}^\infty \alpha_k f_k \quad \text{and} \quad \|z|\mathring{H}^1(\Omega)\|^2 = \sum_{k=1}^\infty |\alpha_k|^2.$$

Using the fact that $Bg = 0$ one can conclude that $(z, g)_{\mathring{H}^1(\Omega)} = 0$. Therefore

$$\sum_{j=1}^{n} \int_{\Omega} \Big| \frac{\partial v}{\partial x_j}(x) \Big|^2 dx = \|z|\mathring{H}^1(\Omega)\|^2 + \|g|\mathring{H}^1(\Omega)\|^2 \geq \|z|\mathring{H}^1(\Omega)\|^2 \tag{5.7.34}$$

One can easily obtain

$$\sqrt{B}\, v = \sum_{k=1}^{\infty} \sqrt{\rho_k} \alpha_k f_k. \tag{5.7.35}$$

Therefore, by (5.7.7) and (5.7.35),

$$\| \operatorname{tr}_\Gamma v|L_2(\Gamma)\|^2 = \sum_{k=1}^{\infty} \rho_k |\alpha_k|^2. \tag{5.7.36}$$

Now applying (5.7.32)-(5.7.36) we obtain

$$\sum_{k=1}^{\infty} (\rho_k - \rho)|\alpha_k|^2 \geq 0.$$

As ρ is the largest eigenvalue we conclude that, for all $k \in \mathbb{N}_0$ such that $\rho_k \neq \rho$, $\alpha_k = 0$. So there is $K \in \mathbb{N}_0$ such that

$$z = \sum_{k=1}^{K} \alpha_k f_k,$$

where f_k, $k = 1, ..., K$, are eigenfunctions associated to ρ. Hence z is also an eigenfunction associated to ρ and, consequently,

$$Bv = \rho z + Bg = \rho z. \tag{5.7.37}$$

Hence

$$(Bv, v)_{\mathring{H}^1(\Omega)} = (\rho z, z + g)_{\mathring{H}^1(\Omega)} = \rho \|z|\mathring{H}^1(\Omega)\|^2$$

and, applying (5.7.7),

$$\| \operatorname{tr}_\Gamma v|L_2(\Gamma)\|^2 = \rho \|z|\mathring{H}^1(\Omega)\|^2. \tag{5.7.38}$$

It follows from (5.7.32), (5.7.34) and (5.7.38) that $g = 0$ and so, by (5.7.33) and (5.7.37), we conclude that v is an eigenfunction associated to ρ.

Step 8: We prove part of *(iv)*, i.e., we prove that there is a function u such that

$$Bu = \rho u \quad \text{and} \quad u(x) > 0, \quad x \in \Omega. \tag{5.7.39}$$

Let v be an eigenfunction associated to ρ. Then (5.7.32) is satisfied for v and, consequently, also for $\overline{v}$ (instead of v). So, by what was proved in the previous step, $\overline{v}$ is also an eigenfunction associated to ρ. Thus $\operatorname{Re} v$ and $\operatorname{Im} v$ are also eigenfunctions, if they are different from zero. So we may assume that v is real. Let us prove that $|v|$ is also an eigenfunction associated to ρ, proving that $|v| \in \overset{\circ}{H}{}^1(\Omega)$ and that (5.7.32) is satisfied when one replaces v by $|v|$. By the property referred in [EE87, p. 220, Proposition 2.6] or [GT01, p. 152, Lemma 7.6], $|v| \in H^1(\Omega)$ and, for all $\alpha \in {\mathbb{N}_0}^n$, $|\alpha| = 1$,

$$D^\alpha |v(x)| = \begin{cases} D^\alpha v(x), & v(x) > 0, \\ 0, & v(x) = 0, \\ -D^\alpha v(x), & v(x) < 0. \end{cases} \tag{5.7.40}$$

It follows immediately from (5.7.40) that

$$\sum_{j=1}^n \int_\Omega \Big|\frac{\partial |v|}{\partial x_j}(x)\Big|^2 dx = \sum_{j=1}^n \int_\Omega \Big|\frac{\partial v}{\partial x_j}(x)\Big|^2 dx. \tag{5.7.41}$$

Let us prove that $|v| \in \overset{\circ}{H}{}^1(\Omega)$. Let $(\varphi_k)_k \subset \mathcal{D}(\Omega)$ converge to v in $H^1(\Omega)$. As $\mathcal{D}(\Omega) \subset H^1(\Omega)$, then, for all $k \in \mathbb{N}$, $|\varphi_k| \in H^1(\Omega)$ and (5.7.40) is satisfied if one replaces v by φ_k. Let us prove that

$$\big\| |v| - |\varphi_k| \big| H^1(\Omega) \big\| \to 0 \quad \text{as} \quad j \to \infty. \tag{5.7.42}$$

One can easily conclude that

$$\big\| |v| - |\varphi_k| \big| L_2(\Omega) \big\|^2 \leq \big\| v - \varphi_k \big| L_2(\Omega) \big\|^2 \to 0 \quad \text{as} \quad j \to \infty.$$

Let us study $\big\| D^\alpha |v| - D^\alpha |\varphi_k| \big| L_2(\Omega) \big\|$, for $\alpha \in \mathbb{N}_0$. The integral $\int_\Omega \big| D^\alpha |v| - D^\alpha |\varphi_k| \big|^2$ is less than or equal to

$$\int\limits_{\{v,\varphi_k>0\}} |D^\alpha v - D^\alpha \varphi_k|^2 + \int\limits_{\{v,\varphi_k<0\}} |-D^\alpha v + D^\alpha \varphi_k|^2 + \int\limits_{\{v=\varphi_k=0\}} 0 \tag{5.7.43}$$

$$+ \int\limits_{\{v>0,\varphi_k\leq 0\}} (|D^\alpha v| + |D^\alpha \varphi_k|)^2 + \int\limits_{\{v<0,\varphi_k\geq 0\}} (|-D^\alpha v| + |-D^\alpha \varphi_k|)^2 + \int\limits_{\{v=0,\varphi_k\neq 0\}} |D^\alpha \varphi_k|^2 \tag{5.7.44}$$

The terms in (5.7.43) can be estimated from above by $\|v - \varphi_k | H^1(\Omega)\|^2$. Let us prove that the first term in (5.7.44) converges to 0 when $k \to \infty$:

$$\begin{aligned}\int\limits_{\{v>0,\varphi_k\leq 0\}} (|D^\alpha v| + |D^\alpha \varphi_k|)^2 &\leq 6 \int\limits_{\{v>0,\varphi_k\leq 0\}} |D^\alpha v|^2 + 4 \int\limits_{\{v>0,\varphi_k\leq 0\}} |D^\alpha \varphi_k - D^\alpha v|^2 \\ &\leq 6 \int\limits_{\{v>0,\varphi_k\leq 0\}} |D^\alpha v|^2 + 4\|\varphi_k - v|H^1(\Omega)\|^2. \qquad (5.7.45)\end{aligned}$$

Let us study

$$\int\limits_{\{v>0,\varphi_k\leq 0\}} |D^\alpha v|^2 = \int\limits_\Omega \chi_{\{v>0,\varphi_k\leq 0\}}(x)|D^\alpha v(x)|^2. \qquad (5.7.46)$$

There exist a subsequence of $(\varphi_k)_k$, which we will still denote by $(\varphi_k)_k$, and $A \subset \mathbb{R}^n$, with $|A| = 0$, such that

$$\varphi_k(x) \to v(x), \quad \text{as} \quad k \to \infty, \quad \text{for all} \quad x \in \Omega \setminus A. \qquad (5.7.47)$$

Let $x \in \Omega \setminus A$. If $v(x) \leq 0$, then $\chi_{\{v>0,\varphi_k\leq 0\}}(x) = 0$. Otherwise, by (5.7.47), for all k sufficiently large, $\varphi_k(x) > 0$ and so $\chi_{\{v>0,\varphi_k\leq 0\}}(x) = 0$. Hence the function $\chi_{\{v>0,\varphi_k\leq 0\}}|D^\alpha v|^2$ converges pointwise almost everywhere to 0. Furthermore, for all $k \in \mathbb{N}$, $\chi_{\{v>0,\varphi_k\leq 0\}}|D^\alpha v|^2$ is less than or equal to $|D^\alpha v|^2$, which is integrable. Therefore, by the Dominated Convergence Theorem (cf. [Lan93, p. 141, Theorem 5.8], for example),

$$\int\limits_\Omega \chi_{\{v>0,\varphi_k\leq 0\}}|D^\alpha v|^2 \to 0 \quad \text{as} \quad k \to \infty. \qquad (5.7.48)$$

By (5.7.45), (5.7.46) and (5.7.48), the first term in (5.7.44) converges to 0 as $k \to \infty$. The same can be proved analogously for the second term in (5.7.44). The third term in (5.7.44) also converges to 0, because by the proof of Lemma 7.6 in [GT01, p. 152], if $v(x) = 0$, then $D^\alpha v(x) = 0$. Thus

$$\int\limits_{\{v=0,\varphi_k\neq 0\}} |D^\alpha \varphi_k|^2 = \int\limits_{\{v=0,\varphi_k\neq 0\}} |D^\alpha \varphi_k - D^\alpha v(x)|^2 \leq \|\varphi_k - v|H^1(\Omega)\|^2,$$

concluding the proof of (5.7.42). Let $\varepsilon > 0$. We fix k sufficiently large such that

$$\big\||v| - |\varphi_k|\big|H^1(\Omega)\big\| < \varepsilon. \qquad (5.7.49)$$

Let $\chi \in \mathcal{D}(\Omega)$ be such that $0 \le \chi \le 1$ and $\chi(x) = 1$ for all x in a neighborhood of $\operatorname{supp} \varphi_k$. Consider

$$c_\chi := \sup_{|\alpha|=1} \sup_{x\in\Omega} |D^\alpha \chi(x)|^2. \tag{5.7.50}$$

As the set of restrictions to Ω of all functions in $C_0^\infty(\mathbb{R}^n)$ is dense in $H^1(\Omega)$ (cf. [EE87, p. 248, Theorem 4.7]), there is $\phi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\big\| |\varphi_k| - \phi \big| H^1(\Omega)\big\| < \frac{\varepsilon}{\max\{1, c_\chi\}}, \tag{5.7.51}$$

for c_χ as in (5.7.50). The function $\chi\phi$ belongs to $\mathcal{D}(\Omega)$. Let us prove that

$$\big\| |v| - \chi\phi \big| H^1(\Omega)\big\| \lesssim \varepsilon. \tag{5.7.52}$$

Let $\alpha \in \mathbb{N}_0{}^n$, with $|\alpha| \le 1$. Then

$$\begin{aligned}\big\|D^\alpha(|\varphi_k| - \chi\phi)\big|L_2(\Omega)\big\|^2 &= \int\limits_{\operatorname{supp}\varphi_k} |D^\alpha|\varphi_k|(x) - D^\alpha\phi(x)|^2 dx + \int\limits_{\Omega\setminus(\operatorname{supp}\varphi_k)} \big|D^\alpha|\varphi_k|(x) - D^\alpha(\chi\phi)(x)\big|^2 dx \\ &\le \big\|D^\alpha(|\varphi_k| - \phi)\big|L_2(\Omega)\big\|^2 + \int\limits_{\Omega\setminus(\operatorname{supp}\varphi_k)} |D^\alpha(\chi\phi)(x)|^2 dx \end{aligned} \tag{5.7.53}$$

Let $|\alpha| = 1$ (the case $|\alpha| = 0$ can be treated analogously). Then

$$\begin{aligned}\int\limits_{\Omega\setminus(\operatorname{supp}\varphi_k)} |D^\alpha(\chi\phi)(x)|^2 dx &\le 2\max\{1, c_\chi\}\Big(\int\limits_{\Omega\setminus(\operatorname{supp}\varphi_k)} |\phi(x)|^2 dx + \int\limits_{\Omega\setminus(\operatorname{supp}\varphi_k)} |D^\alpha\phi(x)|^2 dx\Big) \\ &= 2\max\{1, c_\chi\} \int\limits_{\Omega\setminus(\operatorname{supp}\varphi_k)} \big||\varphi_k(x)| - \phi(x)\big|^2 dx \\ &\quad + 2\max\{1, c_\chi\} \int\limits_{\Omega\setminus(\operatorname{supp}\varphi_k)} \big|D^\alpha|\varphi_k|(x) - D^\alpha\phi(x)\big|^2 dx \\ &\le 4\max\{1, c_\chi\} \big\||\varphi_k| - \phi\big|H^1(\Omega)\big\|^2 \end{aligned} \tag{5.7.54}$$

Now one concludes (5.7.52) by (5.7.49), (5.7.51) and (5.7.53)-(5.7.54) and, therefore, $|v| \in \mathring{H}^1(\Omega)$.

Next we prove that

$$\| \operatorname{tr}_\Gamma v | L_2(\Gamma)\| = \| \operatorname{tr}_\Gamma |v| | L_2(\Gamma)\|. \tag{5.7.55}$$

As v is an eigenfunction, it follows from Step 6, (1.2.4), (2.1.2) and (iv) in Remark 2.1.3 that, for $\delta \in (0, 2-n-\overline{s}(\boldsymbol{h}))$, $v \in \mathcal{C}^{2-n-\overline{s}(\boldsymbol{h})-\delta}(\Omega)$. Hence v is a continuous function in $\overline{\Omega}$ and, consequently, $\operatorname{tr}_\Gamma v = v_{|\Gamma}$. This follows from Proposition 3.4.2 and Theorem 3.4.15 in [Bri01, p. 107 and p. 114]. As v is continuous, $|v|$ is continuous and so $\operatorname{tr}_\Gamma |v| = |v|_{|\Gamma}$ and now (5.7.55) follows immediately.

Therefore, by (5.7.7), (5.7.41) and (5.7.55),

$$\rho \sum_{j=1}^{n} \int_\Omega \Big|\frac{\partial |v|}{\partial x_j}(x)\Big|^2 dx = \int_\Gamma |(\operatorname{tr}_\Gamma |v|)(\gamma)|^2 d\mu(\gamma)$$

and so, by the previous step, $|v|$ is also an eigenfunction associated to ρ and the function $\omega = |v| - v$ satisfies $B\omega = \rho\omega$. As ω is continuous in $\overline{\Omega}$, $\operatorname{tr}_\Gamma \omega = \omega_{|\Gamma} \geq 0$. By what was done in Step 2, there is a set A with Lebesgue measure 0 such that

$$\rho\, \omega(x) = (-\Delta)^{-1} \circ \operatorname{tr}^\Gamma \omega(x) = \int_\Gamma G(x,\gamma)(\operatorname{tr}_\Gamma \omega)(\gamma) d\mu(\gamma), \tag{5.7.56}$$

for all $x \in \Omega \setminus (A \cup \Gamma)$. Let us prove that the equality (5.7.56) holds for all $x \in \Omega$. We present the proof for $n \geq 3$. The case $n = 2$ can be studied similarly. So assume that $n \geq 3$.

First we prove that (5.7.56) holds for all $x \in \Gamma$. We recall that, as $\underline{s}(\boldsymbol{h}) > -n$, then, by Corollary 3.1.11, Γ satisfies the ball condition. Thus, by Definition 3.1.9, there is a number $\eta \in (0,1)$ such that for all $\gamma^0 \in \Gamma$, for all $j \in \mathbb{N}$, there is a ball $B(z_j, \eta 2^{-j})$ contained in $B(\gamma^0, 2^{-j})$ which does not intersect Γ. As $|A| = 0$, there is, for all $j \in \mathbb{N}$, $x_j \in B(z_j, \eta 2^{-j-1}) \setminus A$. Let us fix $\gamma^0 \in \Gamma$ and a sequence $(x_j)_j$ as we have just described. As $(x_j)_j$ converges to γ^0, ω is a continuous function and $(x_j)_j \in \Omega \setminus (A \cup \Gamma)$,

$$\rho\omega(\gamma^0) = \lim_{j\to\infty} \rho\omega(x_j) = \lim_{j\to\infty} \int_\Gamma G(x_j,\gamma)(\operatorname{tr}_\Gamma \omega)(\gamma) d\mu(\gamma). \tag{5.7.57}$$

One can also prove that

$$\lim_{j\to\infty} G(x_j,\gamma)(\operatorname{tr}_\Gamma \omega)(\gamma) = G(\gamma^0,\gamma)(\operatorname{tr}_\Gamma \omega)(\gamma), \quad \text{for all} \quad \gamma \in \Gamma \setminus \{\gamma^0\}. \tag{5.7.58}$$

Let us prove that the functions $G(x_j,\cdot) \operatorname{tr}_\Gamma \omega$ can be estimated from above by a μ-integrable function. Suppose that j and γ are such that $2^{-j+1} \leq |\gamma - \gamma^0|$. Thus

$$|x_j - \gamma| \geq |\gamma - \gamma^0| - |x_j - \gamma^0| \geq |\gamma - \gamma^0| - 2^{-j} \geq 2^{-1}|\gamma - \gamma^0|. \tag{5.7.59}$$

Now we suppose that $2^{-j+1} > |\gamma - \gamma^0|$. Then

$$|x_j - \gamma| \geq \eta 2^{-j-1} > \eta 2^{-2}|\gamma - \gamma^0|. \tag{5.7.60}$$

So, by (5.7.59)-(5.7.60) and by Proposition 5.7.5, as $\mathrm{tr}_\Gamma\, \omega \geq 0$,

$$G(x_j, \gamma)(\mathrm{tr}_\Gamma\, \omega)(\gamma) \lesssim |\gamma^0 - \gamma|^{2-n}(\mathrm{tr}_\Gamma\, \omega)(\gamma), \quad j \in \mathbb{N}, \quad \gamma \in \Gamma. \tag{5.7.61}$$

We prove that the function on the right of (5.7.61) is μ-integrable: let $\delta \in (0, 2-n-\overline{s}(\boldsymbol{h}))$. Then

$$\begin{aligned}\int_\Gamma |\gamma^0 - \gamma|^{2-n}(\mathrm{tr}_\Gamma\, \omega)(\gamma)d\mu(\gamma) \;&\lesssim\; \max_{\lambda\in\Gamma} \omega(\lambda) \sum_{j=0}^{\infty} \int_{c_0 2^{-(j+1)} < |\gamma-\gamma^0| \leq c_0 2^{-j}} |\gamma - \gamma^0|^{2-n} d\mu(\gamma) \\ &\lesssim\; \max_{\lambda\in\Gamma} \omega(\lambda) \sum_{j=0}^{\infty} 2^{-j(2-n-\overline{s}(\boldsymbol{h})-\delta)} \\ &\lesssim\; \max_{\lambda\in\Gamma} \omega(\lambda).\end{aligned} \tag{5.7.62}$$

By (5.7.57)-(5.7.58), (5.7.61)-(5.7.62) and the Dominated Convergence Theorem (cf. [Lan93, p. 141, Theorem 5.8], for example), we conclude that (5.7.56) holds for all $x \in \Gamma$.

Now let $x \in A \setminus \Gamma$. As Γ is closed, $a := \mathrm{dist}(x, \Gamma) > 0$. We consider the open ball $B(x, a/2)$ and an arbitrary sequence $(x_j)_j \subset B(x, a/2)$ convergent to x. The function ω is continuous and, for each $\gamma \in \Gamma$ fixed, the function $G(\cdot, \gamma)$ is continuous. So, as x and x_j, $j \in \mathbb{N}$, are not elements of Γ,

$$\lim_{j\to\infty} G(x_j, \gamma)(\mathrm{tr}_\Gamma\, \omega)(\gamma) = G(x, \gamma)(\mathrm{tr}_\Gamma\, \omega)(\gamma), \quad \text{for all } \gamma \in \Gamma.$$

As $\omega \geq 0$,

$$G(x_j, \gamma)(\mathrm{tr}_\Gamma\, \omega)(\gamma) \lesssim |x_j - \gamma|^{2-n}(\mathrm{tr}_\Gamma\, \omega)(\gamma) \leq 2^{n-2}a^{2-n}(\mathrm{tr}_\Gamma\, \omega)(\gamma),$$

which is independent of j and μ-integrable:

$$\int_\Gamma 2^{n-2}a^{2-n}(\mathrm{tr}_\Gamma\, \omega)(\gamma)d\mu(\gamma) \leq 2^{n-2}a^{2-n} \max_{\lambda\in\Gamma} \omega(\lambda)\mu(\Gamma) < \infty.$$

Again by the Dominated Convergence Theorem and by the continuity of ω one concludes that (5.7.56) holds.

Now we prove that either $\omega > 0$ or $\omega = 0$. We suppose that there is $x \in \Omega$ such that $\omega(x) > 0$. So by (5.7.56), there is $\lambda \in \Gamma$ such that $(\mathrm{tr}_\Gamma\, \omega)(\lambda) > 0$. As ω is continuous and Γ is an h-set, there is a set of positive μ-measure around λ where $\mathrm{tr}_\Gamma\, \omega > 0$. But in this case, as $G(x, \gamma) > 0$ for all $\gamma \in \Gamma \setminus \{x\}$ and $\mathrm{tr}_\Gamma\, \omega \geq 0$, by (5.7.56), $\omega(y) > 0$ for all $y \in \Omega$.

If $\omega > 0$ then v is strictly negative and $u = -v$ is the eigenfunction we were looking for. If $\omega = 0$ then $v \geq 0$. Applying to v the arguments we used for ω we conclude that either $v = 0$ or $v > 0$. But as v is an eigenfunction, we must have $v > 0$ and so $u = v$ is a positive eigenfunction.

Step 9: Let us prove that the largest eigenvalue is simple. That, together with Step 8, will also prove *(iv)*. Assume that ρ is not simple. Let u and v be $\mathring{H}^1(\Omega)$-orthogonal eigenfunctions associated to ρ. By the previous step, the real functions $\mathrm{Re}\, u$, $\mathrm{Im}\, u$, $\mathrm{Re}\, v$ and $\mathrm{Im}\, v$ are also eigenfunctions associated to ρ, if they are different from 0. Assume that in this subset of functions there are two linearly independent functions. Then one can obtain two $\mathring{H}^1(\Omega)$-orthogonal real eigenfunctions associated to ρ.
Now assume that there are not two linearly independent functions in that set of 4 functions. So, in particular, $\mathrm{Re}\, u$ and $\mathrm{Im}\, u$ are linearly dependent. Then, either $\mathrm{Im}\, u = 0$, or there is a real number α such that $\mathrm{Re}\, u = \alpha \mathrm{Im}\, u$. Therefore, either $u = \mathrm{Re}\, u$, or $u = (\alpha + i)\mathrm{Im}\, u$. In both cases there is a complex number $z_1 \neq 0$ such that $z_1 u$ is a real function. Analogously one concludes that there is $z_2 \in \mathbb{C} \setminus \{0\}$ such that $z_2 v$ is a real function. So there are two $\mathring{H}^1(\Omega)$-orthogonal real eigenfunctions associated to ρ. In the previous step it was also proved that a real eigenfunction associated to the largest eigenvalue is strictly positive or strictly negative. Thus there exist a pair of $\mathring{H}^1(\Omega)$-orthogonal strictly positive eigenfunctions associated to ρ. To simplify the notation let us assume that u and v are such functions. Then

$$0 = (\rho v, u)_{\mathring{H}^1(\Omega)} = (Bv, u)_{\mathring{H}^1(\Omega)} = \int_\Gamma (\mathrm{tr}_\Gamma\, v)(\gamma)(\mathrm{tr}_\Gamma\, u)(\gamma)d\mu(\gamma),$$

which is a contradiction. $\square$

5.8 The fractal Dirichlet Laplacian

In this section we study a fractal Dirichlet Laplacian in the context of h-sets. This problem in the context of d-sets was considered by Triebel in [Tri01, Section 20]. We extend part of the results presented there.

In this section we will assume that conditions in Theorem 5.7.7 are satisfied:

Assumption 5.8.1. *Let $h \in \mathbb{H}$ be a strictly increasing function. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $n \geq 2$, and Γ be an h-set such that $\Gamma \subset \Omega$ and*

$$n - 2 < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) < n.$$

We will denote by B the operator given by (5.6.1) *considered acting in $\mathring{H}^1(\Omega)$, as in Theorem 5.7.7.*
Let σ denote the sequence given by

$$\sigma := (1)\boldsymbol{h}^{-1/2}(n)^{-1/2} = (2^j h(2^{-j})^{-1/2} 2^{-\frac{nj}{2}})_{j\in\mathbb{N}_0}. \tag{5.8.1}$$

5.8.1 The spaces $\mathbb{H}^\sigma(\Gamma)$: a scalar product and the dual space

We recall that, according to Definition 3.2.4 and Remark 5.5.6, as $\Gamma \subset \Omega$,

$$\mathbb{H}^\sigma(\Gamma) = \operatorname{tr}_\Gamma H^1(\Omega).$$

By the definitions of tr_Γ and $\mathring{H}^1(\Omega)$, it follows immediately that $\mathbb{H}^\sigma(\Gamma) = \operatorname{tr}_\Gamma \mathring{H}^1(\Omega)$.

Proposition 5.8.2. *The operator*

$$\operatorname{tr}_\Gamma : N(B)^\perp \to \mathbb{H}^\sigma(\Gamma) \tag{5.8.2}$$

is an isometry.

Proof. Step 1: The proof of injectivity and surjectivity relies in (5.7.6). Let us present the proof of surjectivity. Let $f \in \mathbb{H}^\sigma(\Gamma)$. Then there is $g \in \mathring{H}^1(\Omega)$ such that $\operatorname{tr}_\Gamma g = f$. As $g \in \mathring{H}^1(\Omega)$, there exist $u \in N(B)$ and $v \in N(B)^\perp$ for which $g = u + v$. By (5.7.6), $\operatorname{tr}_\Gamma u = 0$ and so $\operatorname{tr}_\Gamma v = \operatorname{tr}_\Gamma g = f$.

Step 2: Let us prove that

$$\| \operatorname{tr}_\Gamma u | \mathbb{H}^\sigma(\Gamma) \| = \| u | \mathring{H}^1(\Omega) \|,$$

for $u \in N(B)^\perp$.

$$\begin{aligned} \| \operatorname{tr}_\Gamma u | \mathbb{H}^\sigma(\Gamma) \| &= \inf\{ \| v | \mathring{H}^1(\Omega) \| : v \in \mathring{H}^1(\Omega),\ \operatorname{tr}_\Gamma v = \operatorname{tr}_\Gamma u \} \\ &= \inf\{ \| v | \mathring{H}^1(\Omega) \| : v = u + \omega,\ \omega \in N(B) \} \\ &= \| u | \mathring{H}^1(\Omega) \|, \end{aligned}$$

where we used the fact that for $v = u + \omega$, with $u \in N(B)^\perp$ and $\omega \in N(B)$,

$$\| v | \mathring{H}^1(\Omega) \|^2 = \| u | \mathring{H}^1(\Omega) \|^2 + \| \omega | \mathring{H}^1(\Omega) \|^2 \geq \| u | \mathring{H}^1(\Omega) \|^2.$$

So, in (5.8.2) we have an isometry. $\square$

Definition 5.8.3. *We define a scalar product in* $\mathbb{H}^\sigma(\Gamma)$ *as follows*

$$(f, g)_{\mathbb{H}^\sigma(\Gamma)} := (\operatorname{tr}_\Gamma^{-1} f, \operatorname{tr}_\Gamma^{-1} g)_{\mathring{H}^1(\Omega)}, \quad f, g \in \mathbb{H}^\sigma(\Gamma),$$

where tr_Γ *is the bijective operator referred in Proposition 5.8.2.*

Definition 5.8.4. *(i) Let*

$$D(\Gamma) := \operatorname{tr}_\Gamma \mathcal{S}(\mathbb{R}^n) \qquad \text{and} \qquad D'(\Gamma) := (D(\Gamma))'.$$

(ii) We define

$$\mathbb{H}^{\sigma^{-1}}(\Gamma) := (\mathbb{H}^\sigma(\Gamma))'$$

with respect to the dual paring $(D(\Gamma), D'(\Gamma))$.

By the Riesz representation theorem there is an isomorphic map

$$\operatorname{id}_H : \mathbb{H}^{\sigma^{-1}}(\Gamma) \to \mathbb{H}^\sigma(\Gamma) \tag{5.8.3}$$

such that, for all $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$,

$$< F, g >= (g, \operatorname{id}_H F)_{\mathbb{H}^\sigma(\Gamma)}, \quad g \in \mathbb{H}^\sigma(\Gamma), \qquad \text{and} \qquad \| F | \mathbb{H}^{\sigma^{-1}}(\Gamma) \| = \| \operatorname{id} F | \mathbb{H}^\sigma(\Gamma) \|. \tag{5.8.4}$$

Based on this, we define a scalar product in $\mathbb{H}^{\sigma^{-1}}(\Gamma)$.

Definition 5.8.5. *We define a scalar product in* $\mathbb{H}^{\sigma^{-1}}(\Gamma)$ *as follows*

$$(F,G)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} := (\mathrm{id}_H F, \mathrm{id}_H G)_{\mathbb{H}^{\sigma}(\Gamma)}, \quad F,G \in \mathbb{H}^{\sigma^{-1}}(\Gamma),$$

where id_H *is as in* (5.8.3)-(5.8.4).

Remark 5.8.6. *By definition it is clear that* $\mathbb{H}^{\sigma}(\Gamma) \subset L_2(\Gamma)$. *Let* $f \in L_2(\Gamma)$. *Then* f *can be interpreted as a functional in* $L_2(\Gamma)$*:*

$$< f, g >= \int_{\Gamma} g(\gamma)\overline{f(\gamma)}d\mu(\gamma), \quad g \in L_2(\Gamma).$$

As $\mathbb{H}^{\sigma}(\Gamma) \subset L_2(\Gamma)$, *then* $f \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$. *So in this sense* $\mathbb{H}^{\sigma}(\Gamma) \subset L_2(\Gamma) \subset \mathbb{H}^{\sigma^{-1}}(\Gamma)$.

5.8.2 The operator B^{Γ} and complete o.n. systems

Proposition 5.8.7. *Consider that conditions in Assumption 5.8.1 are satisfied. Let*

$$B^{\Gamma} := \mathrm{tr}_{\Gamma} \circ B \circ (\mathrm{tr}_{\Gamma})^{-1}. \tag{5.8.5}$$

Then B^{Γ} *is a non-negative compact self-adjoint operator in* $\mathbb{H}^{\sigma}(\Gamma)$ *with null-space* $N(B^{\Gamma}) = \{0\}$. *Furthermore* B^{Γ} *is generated by the sesquilinear form*

$$(B^{\Gamma} f, g)_{\mathbb{H}^{\sigma}(\Gamma)} = \int_{\Gamma} f(\gamma)\overline{g(\gamma)}d\mu(\gamma), \quad f,g \in \mathbb{H}^{\sigma}(\Gamma). \tag{5.8.6}$$

The positive eigenvalues of B *and* B^{Γ} *coincide. Let* $(\rho_k)_{k\in\mathbb{N}}$ *denote the sequence of the positive eigenvalues of* B^{Γ} *repeated according to multiplicity and ordered by decreasing order of their magnitude, as in Theorem 5.7.7. Then there is complete orthonormal system in* $\mathbb{H}^{\sigma}(\Gamma)$, $(u_k)_{k\in\mathbb{N}}$, *such that*

$$B^{\Gamma} u_k = \rho_k u_k, \quad k \in \mathbb{N}. \tag{5.8.7}$$

Proof. Step 1: Let us prove that the image of the operator B is a subset of $N(B)^{\perp}$. Let $f \in \mathring{H}^1(\Omega)$ arbitrary. Let us prove that $Bf \in N(B)^{\perp}$. Consider $g \in N(B)$ arbitrary. Then, by Theorem 5.7.7, $\mathrm{tr}_{\Gamma}\, g = 0$ and $(Bf, g)_{\mathring{H}^1(\Omega)} = 0$.

Step 2: By Theorem 5.7.7, B^Γ is compact operator. Let us prove that it is generated by the sesquilinear form (5.8.6) and, consequently, a non-negative self-adjoint operator. Let $f, g \in \mathbb{H}^\sigma(\Gamma)$. Then, by (5.7.7) and by the previous step,

$$\begin{aligned}(B^\Gamma f, g)_{\mathbb{H}^\sigma(\Gamma)} &= (\operatorname{tr}_\Gamma^{-1} \operatorname{tr}_\Gamma(B \operatorname{tr}_\Gamma^{-1} f), \operatorname{tr}_\Gamma^{-1} g)_{\mathring{H}^1(\Omega)} \\ &= (B \operatorname{tr}_\Gamma^{-1} f, \operatorname{tr}_\Gamma^{-1} g)_{\mathring{H}^1(\Omega)} \\ &= \int_\Gamma f(\gamma)\overline{g(\gamma)} d\mu(\gamma).\end{aligned}$$

Step 3: Let us prove that $N(B^\Gamma) = \{0\}$. Let $f \in \mathbb{H}^\sigma(\Gamma)$ be such that $B^\Gamma f = 0$. Then $\operatorname{tr}_\Gamma(B \operatorname{tr}_\Gamma^{-1} f) = 0$. By Step 1 and Proposition 5.8.2, $B \operatorname{tr}_\Gamma^{-1} f = 0$. By (5.7.6), $\operatorname{tr}_\Gamma \operatorname{tr}_\Gamma^{-1} f = 0$, i.e., $f = 0$.

Step 4: Let us prove that all the positive eigenvalues for B are eigenvalues for B^Γ. The prove of the reverse implication is analogous. Let $\rho > 0$ and $u \in \mathring{H}^1(\Omega)$ be an eigenvalue and an associated eigenfunction, respectively, for B. Then, by Step 1, $u \in N(B)^\perp$. So, for $v = \operatorname{tr}_\Gamma u \in \mathbb{H}^\sigma(\Gamma)$,

$$B^\Gamma v = \operatorname{tr}_\Gamma(Bu) = \rho \operatorname{tr}_\Gamma u = \rho v.$$

By Proposition 5.8.2, as $u \neq 0$ and $u \in N(B)^\perp$, $v \neq 0$ and so v is an eigenfunction associated to ρ for the operator B^Γ.

Step 5: Let us prove the last assertion of the proposition. It is a well-known result from spectral theory that there is an orthonormal sequence of associated eigenfunctions $(u_k)_{k\in\mathbb{N}} \subset \mathbb{H}^\sigma(\Gamma)$. By [TL80, p. 357, Theorem 4.4], $N(B^\Gamma)$ coincides with the orthogonal complement of the subspace generated by $(u_k)_k$, say $M^\perp$. As we have already proved, $N(B^\Gamma) = \{0\}$ and, so, $M = \mathbb{H}^\sigma(\Gamma)$, i.e., $(u_k)_k$ generates $\mathbb{H}^\sigma(\Gamma)$. □

Proposition 5.8.8. *Let $(u_k)_{k\in\mathbb{N}}$ be as in Proposition 5.8.7 and*

$$v_k = \frac{u_k}{\sqrt{\rho_k}}, \quad k \in \mathbb{N}.$$

Then $(v_k)_{k\in\mathbb{N}}$ is a complete orthonormal system in $L_2(\Gamma)$.

Proof. Let us prove that $\{u_k\}_k$ is a complete system in $L_2(\Gamma)$. One can prove, analogously to what was done in [Tri97, p. 6, Theorem 3.8], that $D(\Gamma)$ is dense in $L_2(\Gamma)$. Let $f \in L_2(\Gamma)$

and $\varepsilon > 0$. There is $\varphi \in D(\Gamma)$ such that

$$\|f - \varphi | L_2(\Gamma)\| < \varepsilon/2. \tag{5.8.8}$$

As $\varphi \in D(\Gamma) \subset \mathbb{H}^\sigma(\Gamma)$ and $\{u_k\}_k$ generates $\mathbb{H}^\sigma(\Gamma)$, there are $k_0 \in \mathbb{N}$ and $(\alpha_k)_{k=1}^{k_0}$ such that

$$\Big\|\varphi - \sum_{k=1}^{k_0} \alpha_k u_k | \mathbb{H}^\sigma(\Gamma)\Big\| < \varepsilon/2. \tag{5.8.9}$$

Hence, as $\mathbb{H}^\sigma(\Gamma) \subset L_2(\Gamma)$, it follows from (5.8.8) and (5.8.9) that

$$\Big\|f - \sum_{k=1}^{k_0} \alpha_k u_k | L_2(\Gamma)\Big\| \lesssim \varepsilon.$$

So $\{u_k\}_k$ is a complete system in $L_2(\Gamma)$. It follows from (5.8.6) and (5.8.7) that

$$(u_j, u_k)_{L_2(\Gamma)} = (B^\Gamma u_j, u_k)_{\mathbb{H}^\sigma(\Gamma)} = \rho_j (u_j, u_k)_{\mathbb{H}^\sigma(\Gamma)}$$

and so $(v_k)_{k\in\mathbb{N}}$ is orthonormal in $L_2(\Gamma)$. $\square$

Proposition 5.8.9. *Let $(u_k)_{k\in\mathbb{N}}$ be as in Proposition 5.8.7 and*

$$\omega_k = \frac{u_k}{\rho_k}, \quad k \in \mathbb{N}.$$

Then $(\omega_k)_{k\in\mathbb{N}}$ is a complete orthonormal system in $\mathbb{H}^{\sigma^{-1}}(\Gamma)$.

Proof. By Remark 5.8.6, $u_k \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$ and, for all $g \in \mathbb{H}^\sigma(\Gamma)$,

$$< u_k, g >= \int_\Gamma g(\gamma)\overline{u_k(\gamma)} d\mu(\gamma).$$

Applying (5.8.6) we obtain

$$< u_k, g >= (g, B^\Gamma u_k)_{\mathbb{H}^\sigma(\Gamma)} = (g, \rho_k u_k)_{\mathbb{H}^\sigma(\Gamma)}, \quad g \in \mathbb{H}^\sigma(\Gamma). \tag{5.8.10}$$

So, by (5.8.3)-(5.8.4) and (5.8.10), we conclude that

$$\mathrm{id}_H\, u_k = \rho_k u_k. \tag{5.8.11}$$

Let $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$. For all $\delta > 0$ the exist $k_0 \in \mathbb{N}$ and $(\alpha_k)_{k=1}^{k_0}$ such that

$$\|\,\mathrm{id}_H\, F - \sum_{k=1}^{k_0} \alpha_k \rho_k u_k | \mathbb{H}^\sigma(\Gamma)\| < \delta.$$

Hence, by (5.8.3)-(5.8.4) and (5.8.11),

$$\|F - \sum_{k=1}^{k_0} \alpha_k u_k | \mathbb{H}^{\sigma^{-1}}(\Gamma)\| = \| \mathrm{id}_H F - \sum_{k=1}^{k_0} \alpha_k \, \mathrm{id}_H u_k | \mathbb{H}^{\sigma}(\Gamma)\| < \delta.$$

Furthermore

$$(u_k, u_j)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} = (\mathrm{id}_H u_k, \mathrm{id}_H u_j)_{\mathbb{H}^{\sigma}(\Gamma)} = \rho_k \rho_j (u_k, u_j)_{\mathbb{H}^{\sigma}(\Gamma)}.$$

Therefore $(u_k/\rho_k)_k$ is a complete o.n. system in $\mathbb{H}^{\sigma^{-1}}(\Gamma)$. □

5.8.3 An extension of B^Γ

By Proposition 5.8.8, for all $f \in L_2(\Gamma)$,

$$f = \sum_{k=1}^{\infty} (f, v_k)_{L_2(\Gamma)} v_k \quad \text{in} \quad L_2(\Gamma) \quad \text{with} \quad \|f|L_2(\Gamma)\|^2 = \sum_{k=1}^{\infty} |(f, v_k)_{L_2(\Gamma)}|^2. \tag{5.8.12}$$

We define

$$\widetilde{\sqrt{B^\Gamma}} f := \sum_{k=1}^{\infty} \sqrt{\rho_k} (f, v_k)_{L_2(\Gamma)} v_k = \sum_{k=1}^{\infty} (f, v_k)_{L_2(\Gamma)} u_k, \quad \text{convergence in} \quad \mathbb{H}^{\sigma}(\Gamma), \tag{5.8.13}$$

which converges by (5.8.12), because $(u_k)_k$ is a o.n. system in $\mathbb{H}^{\sigma}(\Gamma)$. The operator

$$\widetilde{\sqrt{B^\Gamma}} : L_2(\Gamma) \to \mathbb{H}^{\sigma}(\Gamma)$$

is well-defined and it is an extension of $\sqrt{B^\Gamma}$. Furthermore, it is isomorphic. Let us prove that it is surjective. Let $g \in \mathbb{H}^{\sigma}(\Gamma)$. Then, as $(u_k)_k$ is a complete o.n. system in $\mathbb{H}^{\sigma}(\Gamma)$,

$$g = \sum_{k=1}^{\infty} (g, u_k)_{\mathbb{H}^{\sigma}(\Gamma)} u_k, \text{ convergence in } \mathbb{H}^{\sigma}(\Gamma) \text{ and } \|g|\mathbb{H}^{\sigma}(\Gamma)\|^2 = \sum_{k=1}^{\infty} |(g, u_k)_{\mathbb{H}^{\sigma}(\Gamma)}|^2. \tag{5.8.14}$$

As $(v_k)_k$ is a complete o.n. system in $L_2(\Gamma)$,

$$\sum_{k=1}^{\infty} (g, u_k)_{\mathbb{H}^{\sigma}(\Gamma)} v_k, \text{ converges in } L_2(\Gamma) \text{ to, say, } f \text{ and } \|f|L_2(\Gamma)\|^2 = \sum_{k=1}^{\infty} |(g, u_k)_{\mathbb{H}^{\sigma}(\Gamma)}|^2. \tag{5.8.15}$$

From (5.8.12), (5.8.13), (5.8.14) and (5.8.15) one concludes that $\widetilde{\sqrt{B^\Gamma}}f = g$.

It follows immediately as a consequence of (5.8.12)-(5.8.13)

$$\|\widetilde{\sqrt{B^\Gamma}}f|\mathbb{H}^\sigma(\Gamma)\| = \|f|L_2(\Gamma)\|, \quad f \in L_2(\Gamma).$$

Let $\widetilde{\sqrt{B^\Gamma}}'$ denote the dual operator of $\widetilde{\sqrt{B^\Gamma}}$. Hence

$$\widetilde{\sqrt{B^\Gamma}}' : \mathbb{H}^{\sigma^{-1}}(\Gamma) \to L_2(\Gamma)$$

is an isomorphic map.

Let us determine an expression for $\widetilde{\sqrt{B^\Gamma}}'F$, with $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$. By Proposition 5.8.9,

$$F = \sum_{j=1}^{\infty} \Big(F, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j} \quad \text{in} \quad \mathbb{H}^{\sigma^{-1}}(\Gamma) \quad \text{and} \quad \|F|\mathbb{H}^{\sigma^{-1}}(\Gamma)\|^2 = \sum_{j=1}^{\infty} \Big|\Big(F, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)}\Big|^2. \tag{5.8.16}$$

Let $g \in L_2(\Gamma)$ be represented by (5.8.12) with g instead of f. Then

$$\begin{aligned}
< \widetilde{\sqrt{B^\Gamma}}'F, g > &= < F, \widetilde{\sqrt{B^\Gamma}}g > \\
&= \Big\langle \sum_{j=1}^{\infty} \Big(F, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j}, \sum_{k=1}^{\infty} \sqrt{\rho_k}(g, v_k)_{L_2(\Gamma)} v_k \Big\rangle \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \Big(F, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \sqrt{\rho_k}(g, v_k)_{L_2(\Gamma)} \Big\langle \frac{u_j}{\rho_j}, v_k \Big\rangle \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \Big(F, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \sqrt{\rho_k}(g, v_k)_{L_2(\Gamma)} \frac{1}{\sqrt{\rho_j}} < v_j, v_k >
\end{aligned}$$

As

$$< v_j, v_k >= \int_\Gamma v_k(\gamma)\overline{v_j(\gamma)}d\mu(\gamma) = (v_j, v_k)_{L_2(\Gamma)},$$

we obtain

$$\begin{aligned}
< \widetilde{\sqrt{B^\Gamma}}'F, g > &= \sum_{j=1}^{\infty} \Big(F, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} (g, v_j)_{L_2(\Gamma)} \\
&= \Big\langle \sum_{j=1}^{\infty} \sqrt{\rho_j}\Big(F, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j}, \sum_{k=1}^{\infty} (g, v_k)_{L_2(\Gamma)} v_k \Big\rangle
\end{aligned}$$

So

$$\widetilde{\sqrt{B^\Gamma}}' F = \sum_{j=1}^{\infty} \sqrt{\rho_j}\Big(F, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j}. \tag{5.8.17}$$

In particular, if $F \in L_2(\Gamma)$, we obtain, by (5.8.17) and Proposition 5.8.8,

$$\begin{aligned}
\widetilde{\sqrt{B^\Gamma}}' F &= \sum_{j=1}^{\infty} \sqrt{\rho_j}\Big(F, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j} \\
&= \sum_{j=1}^{\infty} \sqrt{\rho_j}\Big(\sum_{k=1}^{\infty} (F, v_k)_{L_2(\Gamma)} v_k, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j} \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \sqrt{\rho_j}(F, v_k)_{L_2(\Gamma)} \sqrt{\rho_k}(\omega_k, \omega_j)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j} \\
&= \sum_{j=1}^{\infty} \sqrt{\rho_j}(F, v_j)_{L_2(\Gamma)} \sqrt{\rho_j} \frac{u_j}{\rho_j} \\
&= \sum_{j=1}^{\infty} \sqrt{\rho_j}(F, v_j)_{L_2(\Gamma)} v_j,
\end{aligned}$$

which is, by (5.8.13), $\widetilde{\sqrt{B^\Gamma}} F$.

Therefore

$$\widetilde{B^\Gamma} := \widetilde{\sqrt{B^\Gamma}} \circ \widetilde{\sqrt{B^\Gamma}}' : \mathbb{H}^{\sigma^{-1}}(\Gamma) \to \mathbb{H}^{\sigma}(\Gamma) \tag{5.8.18}$$

is an isomorphic map and it is an extension of B^Γ.

In the next proposition we state what we have just proved on this extension of B^Γ and we relate it with the initial definition of B^Γ as

$$B^\Gamma = \operatorname{tr}_\Gamma \circ (-\Delta)^{-1} \circ \operatorname{id}^\Gamma. \tag{5.8.19}$$

Proposition 5.8.10. *Consider that the conditions in Assumption 5.8.1 are satisfied. Let* B^Γ *be as in* (5.8.5) *and* $\widetilde{B^\Gamma}$ *be as in* (5.8.18) *. Then*

$$\widetilde{B^\Gamma} : \mathbb{H}^{\sigma^{-1}}(\Gamma) \to \mathbb{H}^{\sigma}(\Gamma) \tag{5.8.20}$$

is an isomorphic map and it is an extension of B^Γ*. Moreover, for all* $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$*, there is an element of* $H^{-1}(\Omega)$*, say* $\widetilde{\operatorname{id}^\Gamma} F$*, such that*

$$\widetilde{B^\Gamma} F = \operatorname{tr}_\Gamma \circ (-\Delta)^{-1} \circ \widetilde{\operatorname{id}^\Gamma} F. \tag{5.8.21}$$

Proof. The first part of the proposition was already proved. Let us prove the second part, relating (5.8.18) and (5.8.19). Consider $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$ as in (5.8.16). Let

$$F_N := \sum_{j=1}^{N} \left(F, \frac{u_j}{\rho_j} \right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j}, \quad N \in \mathbb{N}.$$

So

$$F = \lim_{N\to\infty} F_N \quad \text{in} \quad \mathbb{H}^{\sigma^{-1}}(\Gamma) \quad \text{and} \quad \widetilde{B^\Gamma} F = \lim_{N\to\infty} B^\Gamma F_N \quad \text{in} \quad \mathbb{H}^{\sigma}(\Gamma).$$

Hence the sequence $(\mathrm{tr}_\Gamma \circ (-\Delta)^{-1} \circ \mathrm{id}^\Gamma F_N)_{N\in\mathbb{N}}$ is convergent in $\mathbb{H}^\sigma(\Gamma)$. One can easily see that, for all $N \in \mathbb{N}$, $(-\Delta)^{-1} \circ \mathrm{id}^\Gamma F_N$ belongs to the image of the operator B considered acting in $\mathring{H}^1(\Omega)$. So, according to what was done in Step 1 of the proof of Proposition 5.8.7, $(-\Delta)^{-1} \circ \mathrm{id}^\Gamma F_N$ belongs to $N(B)^\perp$, for all $N \in \mathbb{N}$. Thus, by Proposition 5.8.2, the sequence $((-\Delta)^{-1} \circ \mathrm{id}^\Gamma F_N)_{N\in\mathbb{N}}$ is convergent in $\mathring{H}^1(\Omega)$. Applying Theorem 5.4.1 one concludes that the sequence $(\mathrm{id}^\Gamma F_N)_{N\in\mathbb{N}}$ converges in $H^{-1}(\Omega)$ to, say, $\widetilde{\mathrm{id}^\Gamma} F$. So (5.8.21) is satisfied. □

5.8.4 The main result

In the next theorem we state the existence of solution for a fractal Dirichlet problem in the context of h-sets. It was based in a corresponding result for d-sets, which can be found in [Tri01, Theorem 20.7].

Theorem 5.8.11. *Let $h \in \mathbb{H}$ be a strictly increasing function. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $n \geq 2$ and Γ be an h-set with $\Gamma \subset \Omega$. Suppose that*

$$n - 2 < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) < n.$$

Let $g \in \mathbb{H}^\sigma(\Gamma)$, where σ is as in (5.8.1)*. Then the Dirichlet problem*

$$u \in H^1(\Omega), \quad \Delta u(x) = 0 \quad in \quad \Omega \setminus \Gamma, \tag{5.8.22}$$

$$\mathrm{tr}_{\partial\Omega}\, u = 0, \quad \mathrm{tr}_\Gamma\, u = g, \tag{5.8.23}$$

has at least one solution, which can be represented by

$$u = \lim_{N\to\infty} \int_\Gamma G(\cdot,\gamma)F_N(\gamma)d\mu(\gamma), \quad \textit{in } \mathring{H}^1(\Omega), \tag{5.8.24}$$

where

$$F_N := \sum_{j=1}^{N} \Big(F, \frac{u_j}{\rho_j}\Big)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j}, \quad N \in \mathbb{N}, \quad \textit{for} \quad F = \widetilde{B^\Gamma}^{-1} g \tag{5.8.25}$$

and for $(\rho_k)_k$ and $(u_k)_k$ as in Proposition 5.8.7.

Proof. Let us prove that the Dirichlet problem (5.8.22)-(5.8.23) has solution. By Proposition 5.8.10, particularly (5.8.20), as $g \in \mathbb{H}^\sigma(\Gamma)$, then there is $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$ such that $\widetilde{B^\Gamma}F = g$. Let

$$u := (-\Delta)^{-1} \circ \widetilde{\mathrm{id}^\Gamma} F, \tag{5.8.26}$$

where $\widetilde{\mathrm{id}^\Gamma} F$ is as in the proof of Proposition 5.8.10.

By Proposition 5.8.10 and (5.7.14), $u \in \mathring{H}^1(\Omega)$. In particular, $u \in H^1(\Omega)$ and $\mathrm{tr}_{\partial\Omega}\, u = 0$. Moreover, by (5.8.26),

$$\mathrm{tr}_\Gamma\, u = \widetilde{B^\Gamma} F = g$$

and

$$\Delta u = \widetilde{id^\Gamma} F$$

and so $\operatorname{supp} \Delta u \subset \Gamma$. Therefore

$$\Delta u(x) = 0 \quad \text{in} \quad \Omega \setminus \Gamma.$$

The representation in (5.8.24), which can be also obtained in terms of g as

$$F_N = \sum_{j=1}^{N} \big(g, u_j\big)_{\mathbb{H}^\sigma(\Gamma)} \frac{u_j}{\rho_j}, \quad N \in \mathbb{N},$$

follows from (5.7.8) and (5.8.26). $\square$

List of Symbols

A list of various symbols follows along with the page where they are introduced.

Basic notation

Spaces

Most relevant operators

Classes of functions, sequences, indices

Other symbols

Index

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